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CLEVELAND 6, OHIO

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ABSTRACT

The gyration operator which generates a matrix from a given matrix is defined. It is shown that the gyration is related to the concept of combinatorial equivalence. The hybrid matrices of network theory are gyration matrices. The ABCD and hybrid matrices are combinatorially equivalent to their impedance and admittance matrices.

All combinatorially equivalent matrices have the same degree. The gyration operator also preserves the PR property as well as the rank of the hermitian part.

The gyration operators form an involutary Abelian group. The PR property is shown to extend beyond the impedance and admittance matrices of a passive network, and a complete set of PR matrices is given for the description of the network. The relationships between the scattering and gyration operators are detailed.

A complete synthesis procedure, based on the gyration operator is given. Any PR immittance matrix can be synthesized. Three worked examples are included.

Of special significance is a novel concept of enumerating the degree of a rational matrix function.

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CHAPTER I

The Gyration Operator

1.1 INTRODUCTION. In this first chapter, we define an operator termed the gyration. The gyration operates on a given matrix to produce another matrix and is symbolized by the equation

$$\Gamma(A) = B$$

where A is a given matrix, Γ the gyration operator and B is the resultant matrix, termed the <u>gyration</u> of A.

This relationship is sufficient to make the matrices A and B combinatorially equivalent, a term that was coined by A.W. Tucker [11]. The impedance, admittance chain and hybrid matrices of network theory are all combinatorially equivalent. The work of A.W. Tucker emerged from the linear programming field and is applied here to network theory.

It is shown that the gyration operator preserves the passivity property (PR). It is also shown that all combinatorially equivalent matrices

have the same degree. (The degree measures the complexity of a network.) A third invariant of the gyration operator is the rank of the hermitian part, a property of some significance in network theory.

Of special importance in this chapter is a novel method of enumerating the degree of a rational matrix function.

The relations between the scattering operator and the gyration operator are investigated, and various theorems are detailed.

1.2 POLE MULTIPLICITY OF A RATIONAL MATRIX FUNCTION.

In a previous paper Duffin and Hazony [5] studied the properties of the degree of a rational matrix function $F(s)$. It was brought out that the degree may be defined in several equivalent ways. One of these ways concerned the poles of the minors of $F(s)$.

Definition. Let $F(s)$ be a matrix whose elements are rational functions of the complex variable s. Let k be the number of distinct poles that occur in the matrix elements (the pole at

infinity is counted). Then the degree may be defined to be

(1) $\delta F(s) = h_1 + h_2 + \dots + h_k.$

Here h_j is the maximum multiplicity with which the j^{th} pole appears in the minors of $F(s)$.

By "minor" is meant the determinant of F or the determinant of any square submatrix of F. (In addition if the submatrix has no rows or columns we define the empty minor to have the value 1. This convention will be employed in a lemma to follow.) F need not necessarily be square.

It is convenient to have a notation for the multiplicity of a pole at a given point. Thus, if ζ denotes a complex number or the point at infinity let $\delta_\zeta F(s)$ be defined as the maximum multiplicity of the pole at $s = \zeta$ of any minor of the matrix $F(s)$. We term $\delta_\zeta F$ the multiplicity of $F(s)$ at ζ. With this notation we can describe Relation 1 so as to define the degree of F in the form

(2) $\delta F(s) = \sum_\zeta \delta_\zeta F(s).$

Here the summation is over all points of the com-

plex plane including the point at infinity. When the operator δ_ζ is applied to a scalar function, it will not conflict with the usual meaning of multiplicity.

To derive properties of the multiplicity operator δ_ζ, the following lemma is very useful.

Lemma 1. Let M and N be n by n matrices. Then the determinant of their sum is

$$(3) \qquad |M + N| = \sum_{i=0}^{k} |M|_i |N|^i .$$

Here the sum is over all minors $|M|_i$ of M multiplied by the algebraic complementary minor $|N|^i$ of N. Also $|M|_o = 1$, $|N|^o = |N|$, $|M|_k = |M|$, and $|N|^k = 1$, where k is the total number of minor determinants in an n by n matrix. The proof of this result is given in a previous paper [5].

Lemma 2. If $\delta_\zeta G = 0$, then $\delta_\zeta (F + G) = \delta_\zeta F$.

Proof. Let H = F + G; and suppose H', F', and G' denote square submatrices of H, F, and G. Let $\delta_\zeta |F'| = \delta_\zeta (F)$; but $\delta_\zeta |F''| < \delta_\zeta (F)$ if the submatrix F" has fewer rows than F'. Lemma 1 is now applied with M = F' and N = G'. Then it follows

that one term on the right side of Relation 3 is $|F'|$ alone while all other terms contain minors of F with fewer rows than F'. Thus the term $|F'|$ has a greater multiplicity than any other term. Hence,

$$\delta_\zeta(F' + G') = \delta_\zeta|F' + G'| = \delta_\zeta|F'| = \delta_\zeta F$$

This shows that $\delta_\zeta H \geq \delta_\zeta F$. But $F = H - G$, so by a symmetrical argument it follows that $\delta_\zeta F \geq \delta_\zeta H$. Thus $\delta_\zeta F = \delta_\zeta H$, and the proof is complete.

Lemma 3 to follow is stated here for completeness; it will be used in a later paper.

Lemma 3. Let $\phi(s)$ be a scalar function and $F(s)$ be an n by n matrix function. Then

$$\delta_\zeta(\phi F) \leq \delta_\zeta F + n\delta_\zeta \phi \ .$$

Proof. Let $H = \phi F$, and suppose that $\delta_\zeta H = \delta_\zeta |H'|$ for some m by m minor H'. Then

$$\delta_\zeta H = \delta_\zeta|\phi F'| = \delta_\zeta(\phi^m|F'|) \leq \delta_\zeta \phi^m + \delta_\zeta|F'| \ ,$$

but $\delta_\zeta \phi^m = m\delta_\zeta \phi \leq n\delta_\zeta \phi$ and $\delta_\zeta|F'| \leq \delta_\zeta F$, so the proof is complete.

1.3 GYRATION OF A MATRIX.

Given a square matrix A with matrix elements a_{ij}. If $a_{11} \neq 0$, we define a matrix B with matrix elements

$$
(4) \qquad
\begin{aligned}
b_{11} &= 1/a_{11} & & \\
b_{1\xi} &= -a_{1\xi}/a_{11} & \qquad & \xi = 2,3,\ldots \\
b_{\mu 1} &= a_{\mu 1}/a_{11} & \qquad & \mu = 2,3,\ldots \\
b_{\mu\xi} &= a_{\mu\xi}-a_{\mu 1}a_{1\xi}/a_{11} & &
\end{aligned}
$$

The matrix B is termed <u>the gyration of</u> A <u>about pivot</u> a_{11}. This operation of forming a gyration may be denoted by Γ. Thus,

$$B = \Gamma(A) \ .$$

It is clear then from the above definition that $A = \Gamma(B)$. In other words $\Gamma\Gamma = I$. The term gyration was suggested by the fact that a gyrator transforms an impedance matrix A into B.

Let $x\downarrow$ be an arbitrary column vector. Then A defines a transformation $y\downarrow = Ax\downarrow$. In a three dimensional case this stands for

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Clearly,

$$x_1 = \frac{1}{a_{11}} y_1 - \frac{a_{12}}{a_{11}} x_2 - \frac{a_{13}}{a_{11}} x_3 ,$$

so we can eliminate x_1 from the right side of these equations and obtain

$$\begin{pmatrix} x_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{a_{11}} & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ \frac{a_{21}}{a_{11}} & a_{22} - \frac{a_{21}a_{12}}{a_{11}} & a_{23} - \frac{a_{21}a_{13}}{a_{11}} \\ \frac{a_{31}}{a_{11}} & a_{32} - \frac{a_{31}a_{12}}{a_{11}} & a_{33} - \frac{a_{31}a_{13}}{a_{11}} \end{pmatrix} \begin{pmatrix} y_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Note that the last matrix is precisely B, the gyration of A. It follows that Γ be defined as a transposition of x_1 and y_1 and symbolized as

(5) $\Gamma(x_1\ x_2\ x_3\ y_1\ y_2\ y_3) = (y_1\ x_2\ x_3\ x_1\ y_2\ y_3)$.

A similar definition applies to a gyration of A about any other diagonal element a_{ii} as pivot. This gyration may be denoted as Γ_i. Then it is seen from the transposition property of Γ_i that if i, j, k be a permutation of the integers 1, 2, 3,

$$A^{-1} = \Gamma_i(\Gamma_j[\Gamma_k(A)])$$

This gives a rapid way (closely related to Gaussian elimination) of computing the inverse of a matrix. For example, if

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 3 \end{pmatrix}$$

$$\Gamma_1(A) = \begin{pmatrix} 1 & -2 & -1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\Gamma_2[\Gamma_1(A)] = \begin{pmatrix} 5 & -2 & 1 \\ -2 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\Gamma_3\left(\Gamma_2\left[\Gamma_1(A)\right]\right) = \begin{pmatrix} 6 & -3 & 1 \\ -3 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

and the last matrix is A^{-1}. Then it may be said that the gyration of a matrix is a partial inverse of a matrix.

It will be recognized that the gyration operation is precisely that of forming the hybrid matrix, well known in network theory from the analysis of series parallel configurations[7].

A transformation more general than gyration has been introduced by A.W. Tucker [11]. Tucker calls his relation between matrices combinatorial equivalence. It follows from his Theorem 7 that two matrices are combinatorially equivalent if and only if it is possible to pass from one to the other by a finite succession of elementary operations of the following three types:

1. Interchanging of any two rows
2. Interchanging of any two columns
3. A gyration

It is clear from this characterization and the transposition property of gyrations that a

combinatorial equivalence can be described as a permutation of the sequence $(x_1\ x_2\ x_3\ y_1\ y_2\ y_3)$ such as $(y_1\ x_3\ x_2\ y_2\ y_3\ x_1)$.

A basic property of combinatorial equivalence is given in Tucker's Theorem 3 which follows:

Let $\{\alpha\}$ _denote the set of square submatrices of_ A _of all orders (including an empty submatrix_ ϕ _of order zero). Let_ $\{\beta\}$ _denote the like set for_ B _where_ B _is combinatorially equivalent to_ A. _Then there is a one-to-one correspondence_ $\beta \leftrightarrow \alpha$ _between_ $\{\beta\}$ _and_ $\{\alpha\}$ _such that corresponding subdeterminants_ $|\beta|$ _and_ $|\alpha|$ _are proportional within sign. Specifically_

$$|\beta| = \pm |\alpha| / |\alpha^*| ,$$

where $\alpha = \alpha^*$ _corresponds to_ $\beta = \phi$ _(taking_ $|\phi| = 1$). _The nonsingular matrix_ α^* _is called the pivot of the transformation of_ A _into_ B.

Tucker says that his work on combinatorial equivalence was suggested by the well known simplex method devised by G. B. Dantzig for the solution of linear program problems. We wish to show in

this chapter that combinatorial equivalence has application in the entirely different field of network synthesis. It is worth noting that ideas similar to combinatorial equivalence have been applied to network algebra problems by Bott and Duffin [2].

We now apply our results on the degree of rational matrix functions together with Tucker's Theorem 3.

Theorem 1. Let A be a rational matrix function of the complex variable s and suppose B is combinatorially equivalent to A. Then B is a rational matrix function of the same degree as A.

Proof. Let p denote $|\alpha^*|$, the pivot. Then according to Tucker's theorem $|\beta| = |\alpha| p^{-1}$. Let x be neither a pole of p or p^{-1}, and apply the multiplicity operator δ_x. It follows that $\delta_x|\beta| = \delta_x(|\alpha| p^{-1}) = \delta_x|\alpha|$. Then

$$\delta_x A = \delta_x B$$

Let y be a pole of p. Then it follows that $\delta_y|\beta| \geq \delta_y|\alpha| - \delta_y p$, and so by the definition of

the multiplicity operator

$$\delta_y B \geqq \delta_y A - \delta_y p \ .$$

Let z be a pole of p^{-1}. If $\delta_z A > 0$, let α be the submatrix so that $\delta_z|\alpha| = \delta_z A$. But if $\delta_z A = 0$, let α be the empty matrix, so $|\alpha| = 1$. Then in either case

$$\delta_z|\beta| = \delta_z|\alpha| + \delta_z p^{-1} = \delta_z A + \delta_z p^{-1}.$$

It follows that

$$\delta_z B \geqq \delta_z A + \delta_z p^{-1} \ .$$

The degree of B is defined as $\delta B = \sum_s \delta_s B$, where the summation is over all finite values and also the point at infinity. Clearly the points x, y, and z are distinct but together include all s points. Thus

$$\delta B = \sum_x \delta_x B + \sum_z \delta_z B + \sum_y \delta_y B \ .$$

Substituting the relations just found for $\delta_x B$, $\delta_y B$, and $\delta_z B$ gives

$$\delta B \geqq \sum_x \delta_x A + \sum_y \delta_y A + \sum_z \delta_z A - \sum_y \delta_y p + \sum_z \delta_z p^{-1} \ .$$

The last two sums cancel since $\delta p = \delta p^{-1}$. Thus it follows that $\delta B \geqq \delta A$. But combinatorial equivalence is a symmetric relationship so $\delta A \geqq \delta B$. This proves that $\delta A = \delta B$.

1.4 CHARACTERIZATION OF GYRATION MATRICES.

Let A be an n by n matrix of real numbers, let $x\downarrow$ be a vector, and let $y\downarrow = Ax\downarrow$. The scalar product of the vectors $x\downarrow$ and $y\downarrow$ is denoted by $\vec{x}\, y\downarrow$ and defined as

$$\vec{x}\, y\downarrow = \sum_{1}^{n} x_i y_i \ .$$

If $\vec{x}Ax\downarrow$ is positive for every real vector $\vec{x} \neq \vec{\phi}$, then A is said to be positive definite. This differs from the standard definition in that we are not requiring A to be symmetric. We propose to extend this definition in a natural way when A has complex matrix elements. Let $\vec{x}$ be an arbitrary complex vector and let $\vec{x}^*$ be the complex conjugate. Then if $\vec{x}^*Ax\downarrow$ is in the right half-plane or on the j-axis, we shall say that A is is right definite. This condition may be stated as an inequality

$$\mathrm{Re}\ \vec{x}^*Ax\downarrow \geqq 0 \ .$$

Note that this condition is less restrictive than the condition $\overrightarrow{x}^*Ax\!\downarrow \geq 0$, which is one of the defining conditions for non-negative definite hermitian matrices.

Theorem 2. The gyration of a right definite matrix A is a right definite matrix B.

Proof. It is assumed, of course, that the pivot, say a_{11}, does not vanish. Let $u\!\downarrow$ be an arbitrary vector and let $v\!\downarrow = Bu\!\downarrow$. Let the vector $x\!\downarrow$ be defined as follows:

$$x_1 = v_1, x_2 = u_2, \; x_3 = u_3, \; \ldots, \; x_n = u_n \; .$$

If $y\!\downarrow = Ax\!\downarrow$, then by the property of a gyration $y_1 = u_1$, $y_2 = v_2$, $y_n = v_n$. Thus

$$\overrightarrow{u}^*v\!\downarrow = \overrightarrow{x}^*y\!\downarrow - x_1^*y_1 + x_1y_1^*$$

Taking the real part gives

$$\mathrm{Re}\ \overrightarrow{u}^*v\!\downarrow = \mathrm{Re}\ \overrightarrow{x}^*y\!\downarrow \quad \text{or}$$

$$\mathrm{Re}\ \overrightarrow{u}^*Bu\!\downarrow = \mathrm{Re}\ \overrightarrow{x}^*Ax\!\downarrow \geq 0 \; .$$

This completes the proof.

Definition. A matrix function $F(s)$ of the complex variable s is said to be positive real (PR) if I and II are satisfied:

I. The matrix elements $f_{ij}(s)$ are rational functions of s with real coefficients.

II. For any choice of complex numbers

$x_1, \ldots, x_n$

$$\mathrm{Re} \sum_{1}^{n} \sum_{1}^{n} f_{ij} x_j x_i^* \geq 0 \quad \text{for } \mathrm{Re}\, s \geq 0 .$$

Condition II states that F is a right definite matrix for s in the right half plane. Of course Condition II is not meaningful at a pole.

Let $f(s) = \vec{x}^* F x|$. Then, as is shown in Appendix 1, Condition II is equivalent to the condition:

II a. $\mathrm{Re}\, f(s) \geq 0$ for $\mathrm{Re}\, s = 0$

II b. $f(s)$ has no poles for $\mathrm{Re}\, s > 0$.

II c. For $\mathrm{Re}\, s = 0$ poles of $f(s)$ are simple and have non-negative residues.

The impedance (or admittance) matrix of a passive network is shown Appendix 1 to be positive real. If the network has no gyrators it is reciprocal; i.e., the matrix F satisfies the

symmetry conditions:

III. $f_{ij} = f_{ji}$ <u>for</u> i, j = 1, 2, ...,

We are concerned in this chapter mainly with reciprocal networks. Thus, let G be the gyration of a matrix F, which satisfies I, II, and III and f_{11} does not vanish identically. Then G may be regarded as the hybrid matrix of a reciprocal network. G satisfies I. By Theorem 2, G satisfies II, and by Equation 4 which defines a gyration, the symmetry Condition III is replaced by:

$$\text{III}^1. \quad g_{\mu\xi} = g_{\xi\mu}\,,$$

$$g_{\mu 1} = -g_{1\mu} \qquad \mu,\xi = 2,\ 3,\ \ldots,\ n$$

Condition III^1 may be termed (first) <u>hybrid symmetry</u>.

Note that if G has hybrid symmetry and $g_{11}(s)$ does not vanish identically, then $\Gamma(G)$ has regular symmetry. Thus we may state the following:

<u>Theorem 3</u>. <u>Conditions</u> I, II, <u>and</u> III^1 <u>are necessary and sufficient that</u> G <u>be the (first) gyration of the impedance matrix of a passive network without gyrators, provided</u> $g_{11}(s)$ <u>does not</u>

vanish identically.

Proof. This is a consequence of Theorem 2 and the above definitions.

A positive real matrix F will be said to be IPR if

$$\text{IV.} \qquad \operatorname{Re} \sum_{1}^{n} \sum_{1}^{n} f_{ij} x_j x_i^* = 0 \text{ for } \operatorname{Re} s = 0$$

As is well known, an L C network (i.e., a network without resistors or gyrators) is characterized by having an impedance (or admittance) matrix F satisfying I, II, III, IV.

Theorem 4. Conditions I, II, III^1, and IV are necessary and sufficient that G be the (first) gyration of the impedance matrix of an L C network, provided $g_{11}(s)$ does not vanish identically.

Proof. Let $F = \Gamma(G)$ and let Re s = 0. Then given a vector $\vec{u}$ we have shown in the proof of Theorem 2, that there is a vector $\vec{x}$ so that $\operatorname{Re} \vec{x}^* F x\rangle = \operatorname{Re} \vec{u}^* G u\rangle$ Conversely, given an arbitrary $\vec{x}$ there exists a $\vec{u}$, which satisfies this equation. Then $\operatorname{Re} \vec{u}^* G u\rangle = 0$ for all $\vec{u}$ if and only if $\operatorname{Re} \vec{x}^* F x\rangle = 0$ for all $\vec{x}$. But the latter condition characterizes L C networks,

so the proof is complete.

It is known that matrix elements of an IPR impedance matrix are odd functions of s. Likewise it can be seen from Equation 4 that if G is a hybrid IPR matrix, g_{11} and $g_{\mu\xi}$ are odd functions while $g_{\mu 1}$ and $g_{1\xi}$ are even functions.

If a matrix G satisfies Conditions I, II, and III[1], then G is a positive real matrix function with hybrid symmetry. However, if $g_{11}(s)$ vanishes identically, then it is not the gyration about the first pivot of a symmetric positive real matrix.

1.5 REMOVAL OF IMAGINARY AXIS POLES.

Let G(s) be the hybrid matrix of a passive reciprocal network, and suppose that G has poles on the j axis. According to Condition II c, the function

$$g(s) = \sum_{1}^{n} \sum_{1}^{n} g_{ij}(s) x_i x_j^*$$

can have only simple poles on the j axis. Since the numbers x_i are arbitrary, it is seen that $g_{ij}(s)$ can have at most simple poles on the j axis.

The proof goes as follows:

First take $x_i = \delta_{ih}$ where δ_{ih} is the Kronecker delta. Then we see that $g(s) = g_{hh}(s)$ and so $g_{hh}(s)$ can have only simple poles (by II c). Next, take $x_i = \delta_{i1} + j\delta_{ih}$ and so

$$g(s) = g_{11}(s) + g_{hh}(s) - j[g_{1h}(s) - g_{h1}(s)]$$

Making use of the hybrid symmetry gives

$$g(s) = g_{11}(s) + g_{hh}(s) + 2jg_{h1}(s)$$

Since g, g_{11}, and g_{hh} have at most first-order poles on the j axis, it follows that g_{h1} has a first order pole at most.

Next take $x_i = \delta_{ik} + \delta_{ih}$ for $k \geq 2$, $h \geq 2$. Then similar considerations show that g_{kh} has a first-order pole at most and the proof is complete.

Thus consider a pole at the point $s = s_o = j\omega_o$ on the j-axis and suppose $s_o \neq 0$. Then

$$g_{ij}(s) = \frac{a_{ij}}{s - s_o} + g'_{ij}(s)$$

where a_{ij} is the residue constant and $g'_{ij}(s)$ is

bounded at $s = s_o$. Let $s - s_o = \xi$ be a number in the right half plane. Then for ξ of small absolute value, it is seen the first term on the right is dominant. Consequently by Condition III[1] it is seen that the matrix A has hybrid symmetry. Then for arbitrary $\vec{x}$

$$\vec{x}^*G\vec{x} = \frac{\vec{x}^*A\vec{x}}{s - s_o} + \vec{x}^*G'\vec{x}$$

The first term on the right is dominant for $|\xi|$ small. Hence, it follows from Condition II c that $\vec{x}^*A\vec{x} \geq 0$. In particular, $\vec{x}^*A\vec{x}$ is real.

By Condition I it follows that there is a pole at $s = s_o^* = -j\omega_o$ and

$$g_{ij}(s) = \frac{a_{ij}}{s - s_o} + \frac{a_{ij}^*}{s - s_o^*} + g''_{ij}(s) ,$$

where $g''_{ij}(s)$ is bounded at $s = \pm j\omega_o$. Let

$$p_{ij}(s) = \frac{a_{ij}}{s - s_o} + \frac{a_{ij}^*}{s - s_o^*} .$$

Of course, A* satisfies the same condition as A. Thus P satisfies Conditions I and III[1]. However,

since $\overrightarrow{x^*}Ax\!\downarrow$ and $\overrightarrow{x^*}A^*x\!\downarrow$ are both real non-negative it follows that the quadratic form $\overrightarrow{x^*}Px\!\downarrow$ is a sum of poles on the j axis with real non-negative residues. Thus P satisfies both II and IV, and so *P is hybrid IPR.* The matrix $G'' = G - P$, so clearly I and III[1] are satisfied. II a is satisfied because $\mathrm{Re}\ \overrightarrow{x^*}G''x\!\downarrow = \mathrm{Re}\ \overrightarrow{x^*}Gx\!\downarrow$ on the j axis. II b is satisfied because neither G nor P has poles in the right half plane. Note that G" has no poles at $s = \pm j\omega_o$ so at any other j-axis point, residue residue$(\overrightarrow{x^*}G''x\!\downarrow)$ = residue$(\overrightarrow{x^*}Gx\!\downarrow)$. Hence, G" satisfies II c. *This proves that G" is a hybrid PR matrix.*

Theorem 5. If G *is a hybrid* PR *matrix then*

$$G = P + Q$$

where P *is a hybrid* IPR *matrix and* Q *is a hybrid* PR *matrix without poles on the* j-axis.

Proof. If the only poles of G are at $\pm j\omega_o$, then the above argument gives the proof with $Q = G''$. Otherwise the process is repeated on G", etc. It is seen that a pole at zero or infinity can be removed by a similar method.

1.6 SCATTERING MATRICES AND HYBRID MATRICES [3]

Let A be a matrix such that $(A + I)^{-1}$ exists. Then the scattering operator S is defined as

$$S(A) = (A - I)(A + I)^{-1} .$$

Theorem 6. A *is a right definite matrix if and only if* $T = I - S_t^*S$ *is a non-negative definite hermitian matrix* (where S_t^* is the conjugate transpose of S, and S stands for S(A)).

Proof. Given $x\downarrow$ let $y\downarrow = (A + I)^{-1}x\downarrow$ so $x\downarrow = (A + I)y\downarrow$. Then

$$\overrightarrow{x^*}Tx\downarrow = \overrightarrow{x^*}x\downarrow - \overrightarrow{x^*}S_t^*Sx\downarrow$$

$$= \overrightarrow{y^*}(A_t^*+I)(A+I)y\downarrow - \overrightarrow{y^*}(A_t^*-I)(A-I)y\downarrow$$

$$= 2\overrightarrow{y^*}Ay\downarrow + 2\overrightarrow{y}A^*y^*\downarrow$$

Then $\overrightarrow{x^*}Tx\downarrow = 4\,\mathrm{Re}\,(\overrightarrow{y^*}Ay\downarrow)$ *and the proof is complete.*

It is desired to find the scattering matrix S(B) of the gyration matrix $B = \Gamma(A)$. It is sufficient to consider the case of three-by-three matrices. First it is clear that

B =

$$\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11}^{-1} & -a_{12}a_{11}^{-1} & -a_{13}a_{11}^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This can be written in the form (omitting zeros)

(6) B =

$$\left\{\begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} A + \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}\right\} \left\{\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} A + \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}\right\}^{-1}$$

Moreover B + I =

$$\left\{\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} A + \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\right\} \left\{\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} A + \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}\right\}^{-1}$$

and B - I =

$$\left\{\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} A - \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\right\} \left\{\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} A + \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}\right\}^{-1}$$

Since $(A + I)^{-1}$ is assumed to exist, it follows that $(B + I)^{-1}$ exists. This shows that

$$S(B) = (B - I)(B + I)^{-1}$$

$$= \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} (A - I)(A + I)^{-1}.$$ This proves

the following:

Theorem 7. _Let_ A _be a matrix such that_ $a_{11} \neq 0$ _and_ $(A + I)^{-1}$ _exists._ _Let_ B _be the gyration of_ A. _Then_ $S(B) = JS(A)$, _where_ $S(A)$ _is the scattering matrix of_ A, $S(B)$ _is the scattering matrix of_ B, _and_ J _is the identity except_ $j_{11} = -1$.

Now it is noted that $J^2 = I$ so we have

Corollary: $T(B) = T(A)$ _so_ B _is_ PR _if and only if_ A _is_ PR.

Now suppose that A is a rational matrix function.

Theorem 8. _If_ A _is a rational matrix function and_ $(A + I)^{-1}$ _exists, then the degree of_ A _and the degree of the scattering matrix_ $S(A)$ _are equal_.

Proof. Since $S(A) = I - 2(A + I)^{-1}$, we see that $\delta S = \delta(A + I)^{-1} = \delta(A + I) = \delta A$.

Corollary: _The degree of_ A _is equal to the degree of_ B.

Proof. $S(B) = JS(A)$ and since J is a nonsingular constant matrix, $\delta S(B) = \delta S(A)$. Thus $\delta B = \delta A$.

Theorem 9. Let p, q be any two complex numoers for which $B(p)$, $B(q)$ are defined, and where $B = \Gamma(A)$. Then

$$B(p) + B(q)_T = M(q)_T[A(p) + A(q)_T]M(p),$$

where
$$M(p) = \begin{pmatrix} \frac{1}{a_{11}(p)} & \frac{-a_{12}(p)}{a_{11}(p)} & \frac{-a_{13}(p)}{a_{11}(p)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proof. By Equation 6 above,

$B = CD^{-1}$ where

$$C = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} A + \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad \text{and}$$

$$D = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} A + \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Hence

$$B(p) + B(q)_T = C(p)D^{-1}(p) + [C(q)D^{-1}(q)]_T$$

$$= D^{-1}(q)_T [D(q)_T C(p) + C(q)_T D(p)] D^{-1}(p)$$

Now

$$D(q)_T C(p) + C(q)_T D(p) = A(p) + A(q)_T$$

also

$$D^{-1}(p) = \begin{pmatrix} a_{11}(p) & a_{12}(p) & a_{13}(p) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{a_{11}(p)} & \frac{-a_{12}(p)}{a_{11}(p)} & \frac{-a_{13}(p)}{a_{11}(p)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Expanding $D^{-1}(q)$ in a similar fashion completes the proof of the theorem.

Corollary. Let $B = \Gamma(A)$ be defined. Then the rank of the hermitian part of B is the same as the rank of the hermitian part of A.

Proof. Set $p = s$ and $q = s^*$ in the above theorem. The invariance of rank under a gyration extends

also to the parahermitian part of the matrix A; i.e., $\frac{1}{2}[A(s) + A(-s)_T]$, subject to a similar set of assumptions as in the above corollary.

CHAPTER II

The Gyration Operator in Networks.

2.1 INTRODUCTION. In this chapter, the Γ operator is applied to network synthesis and a complete procedure is given for the realization of symmetric PR impedance (or admittance) matrices.

2.2 HYBRID MATRICES IN NETWORK ANALYSIS.

Suppose we are given a network whose impedance matrix is Z(s). Then the currents and voltages are related by

$$V\downarrow = Z\ I\downarrow$$

Let $H = \Gamma(Z)$ where it is assumed for the moment that $\Gamma(Z)$ is defined. Partitioning $V\downarrow$ and $I\downarrow$ as follows

$$V\downarrow = \begin{pmatrix} v_1 \\ \hline V_2\downarrow \end{pmatrix}$$

$$I\downarrow = \begin{pmatrix} i_1 \\ \hline I_2\downarrow \end{pmatrix}$$

we have by Equation 5,

$$\Gamma[\,v_1 \; V_2{\downarrow} \; i_1 \; I_2{\downarrow}\,] = [\,i_1 \; V_2{\downarrow} \; v_1 \; I_2{\downarrow}\,]$$

i.e.
$$\begin{pmatrix} i_1 \\ \hline V_2{\downarrow} \end{pmatrix} = H \begin{pmatrix} v_1 \\ \hline I_2{\downarrow} \end{pmatrix} .$$

Consider the impedance Z' and Z" connected as shown in Figure 1 with ports 1 in parallel and ports 2 through n in series.*

By definition

$$V'{\downarrow} = Z'I'{\downarrow}$$

$$V''{\downarrow} = Z''I''{\downarrow}$$

Letting $H' = \Gamma(Z')$ and $H'' = \Gamma(Z'')$ we have

*It is assumed that this connection does not change the immittance properties of either network. This can always be ensured by the appropriate use of isolation transformers.

$$\begin{pmatrix} i_1' \\ \hline V_2'\downarrow \end{pmatrix} = H' \begin{pmatrix} v_1' \\ \hline I_2'\downarrow \end{pmatrix}$$

$$\begin{pmatrix} i_1'' \\ \hline V_2''\downarrow \end{pmatrix} = H'' \begin{pmatrix} v_1'' \\ \hline I_2''\downarrow \end{pmatrix}$$

But $v_1' = v_1''$ and $I_2'\downarrow = I_2''\downarrow$.

Hence,

$$\begin{pmatrix} i_1' + i_1'' \\ \hline V_2'\downarrow + V_2''\downarrow \end{pmatrix} = (H' + H'') \begin{pmatrix} v_1' \\ \hline I_2''\downarrow \end{pmatrix}$$

We have thus proved the following:

Theorem 10. Let two networks be connected with ports 1 in parallel and ports 2 through n in series. Then the hybrid matrix for the over-all network equals the sum of the hybrid matrices for the individual networks.

2.3 SYNTHESIS PROCEDURE.

The network synthesis problem consists of associating a passive electrical network with a prescribed PR matrix function. The basic approach

is to extract simple sections from the matrix that can be realized by inspection, thereby reducing the degree of the matrix. When the degree is reduced to zero, the process terminates. These simple sections are extracted in such a way as to ensure that what remains is still PR and hence realizable.

Assume that we are given a symmetric PR impedance matrix $\hat{Z}$ to be synthesized. As a first step, all obvious imaginary axis poles are removed. These are readily synthesized. Once this is completed, various schemes are used to induce further imaginary axis poles, which are removed and synthesized. One of the methods of inducing these further poles in the scalar case is due to Brune [12].

In what follows, we shall use the Γ operator to give a new extension of the Brune synthesis to n port. Such extensions already have been made by a number of people. Brockway McMillan [8] proceeded with the removal of a certain amount of resistance from each port of $\hat{Z}$ until the even-part matrix Ev $\hat{Z}(s) = \frac{1}{2}[\hat{Z}(s) + \hat{Z}(-s)]$ becomes

singular at some point on the j axis, say $j\omega_o$. B.D.H. Tellegen [10] has shown in his method that Ev $\hat{Z}$ can be made singular by the removal of resistance from just one port.

Once Ev $\hat{Z}(s)$ is singular, the odd part then is appropriately modified at $j\omega_o$ resulting in $\hat{Z}(j\omega_o)$ being singular. Hence, $\hat{Z}(j\omega_o)^{-1}$ has a pole that can be removed. V. Belevitch [1] considers the Brune methods of the above two and extends their results to nonreciprocal networks. R.W. Newcomb [9] considers the nonreciprocal case with additional detail. (The methods of Gewertz and Oono [6, p. 276], both non-Brune, consist of removal of j-axis poles accompanied by successive matrix inversions. A similar method given by Duffin [4] shows that network synthesis can be viewed as a purely algebraic process).

Our extension of the Brune method differs from the above in that it is not necessary at any stage to invert a matrix. As will be shown, the method is minimal in the sense of the following theorem, stated by Tellegen [10]:

Minimal Theorem. A positive real symmetric matrix function of degree d can be synthesized as the impedance matrix of a network having a total of d reactive elements. By a reactive element we mean either an inductor or a capacitor. An ideal transformer is not regarded as a reactive element.

One of the results of the work to follow is a new proof of the Minimal Theorem.

Following the above procedures, we assume that Ev $\hat{Z}(j\omega_o)$ is singular. It is shown in [6] that there exists a real constant matrix A such that

$$\tilde{Z} = A_T \hat{Z} A$$

has Re $\tilde{z}_{11}(j\omega_o) = 0$.

(In order to compensate for the application of this congruence transformation, we also apply the inverse congruence transformation, which may be realized by ideal transformers as shown in [6]).

If $\tilde{z}_{11}$ is identically zero then it is shown in [6] that the entire first row and first column of $\tilde{Z}$ is identically zero. Synthesis then resumes on the rest of $\tilde{Z}$. We can thus assume without loss

of generality that $\tilde{z}_{11}$ is not identically zero.

Suppose Re $\tilde{z}_{11}$ is identically zero on the j-axis. Then $\tilde{z}_{11}$ is IPR, and since it cannot have j-axis poles (these have all been removed) it must be that $\tilde{z}_{11}$ is identically zero. Since this has already been ruled out, we may take it that Re $\tilde{z}_{11}$ is not identically zero on the j-axis.

Finally, suppose $\tilde{Z}$ is identically singular. Then it is shown in [6] that there exists a congruence transformation such that $C_T \tilde{Z} C$ has its entire first row and first column identically zero. As before, we may assume without loss of generality that this is not the case, i.e., $\tilde{Z}$ is not identically singular.

Assume then that $\tilde{Z}$ is an n x n PR impedance matrix without j-axis poles, not identically singular, $\tilde{z}_{11}$ is not identically zero, Re $\tilde{z}_{11}$ is not identically zero on the j-axis and Re $\tilde{z}_{11} = 0$ at $j\omega_o$.

Following the classical Brune tradition [12] we now add a scalar $b = sL$ or $1/sC$ to $\tilde{z}_{11}$ so that $\tilde{z}_{11} + b$ is zero at $j\omega_o$. If $\omega_o = 0$ or ∞ then Re $\tilde{z}_{11}(j\omega_o) = 0$ implies that $\tilde{z}_{11}(j\omega_o) = 0$ and no

impedance b is required.

Let $Z = \tilde{Z} + \left(\begin{array}{c|c} b & \vec{\phi} \\ \hline \phi| & \phi \end{array}\right)$

Then $z_{11}(j\omega_o) = 0$ but z_{11} is not identically zero. Since $\tilde{Z}$ has no j-axis poles, it follows that Z has only the j-axis pole possibly due to b. Hence if $0 < \omega_o < \infty$, then Z has at most one j-axis pole in only z_{11} and this pole is either at 0 or ∞, depending on b. If $\omega_o = 0$ or ∞, Z has no j-axis poles.

We are now in a position to reduce the degree of Z by the removal of a lossless section.

Theorem 11. Let $\tilde{Z}(s)$ be an n x n PR impedance matrix a) not identically singular, b) without j-axis poles, c) with Re $\tilde{z}_{11}$ not identically zero on the j-axis and d) Re $\tilde{z}_{11} = 0$ at $j\omega_o$.

Let $Z = \tilde{Z} + \left(\begin{array}{c|c} b & \vec{\phi} \\ \hline \phi| & \phi \end{array}\right)$ where the IPR scalar b is so chosen that $z_{11}(j\omega_o) = 0$.

Then Z may be decomposed in the series parallel manner shown in Figure 1, where

1. Z' is PR and $\delta Z' = \delta Z - \delta Z''$.
2. Z'' is IPR and of degree 1 or 2, and
3. Z'' contains at most 1 inductor, 1 capacitor and 1 n-port ideal transformer (1 core, n windings).

(These are referred to as propositions 1, 2, and 3 respectively). Proof is deferred.

This theorem is the basis of the procedure. By repeated application of this threorem together with Theorem 12 to follow we eventually reduce the degree of $\hat{Z}$ to zero at which point the iteration terminates.

To apply this theorem at each cycle, we may have to add a reactance to $\tilde{z}_{11}$. At the completion of the cycle we therefore have to subtract this reactance. It turns out, just as in the scalar case, that this subsequent negative element can be incorporated in a perfectly coupled transformer.

Theorem 12. Let Z, $\tilde{Z}$ and b be as defined in Theorem 11. Then after the application of that theorem to split Z, it is always possible to incorporate the compensating negative reactor into

a perfectly coupled transformer.

The proof is again deferred. The entire synthesis procedure is displayed in flow chart form in Figure 2.

Example 1

For the sake of clarity, we choose an example that is already tailored appropriately for direct application of Theorem 11. Let Z(s) be given by

$$Z = \begin{pmatrix} \dfrac{s^2 + 1}{s^2 + s + 1} & \dfrac{1}{s^2 + s + 1} \\[3em] \dfrac{1}{s^2 + s + 1} & \dfrac{2(s + 1)}{s^2 + s + 1} \end{pmatrix}$$

Then $\Gamma(Z) =$

$$\begin{pmatrix} \dfrac{s^2+s+1}{s^2+1} & \dfrac{-1}{s^2+1} \\[3em] \dfrac{1}{s^2+1} & \dfrac{2(s+1)}{s^2+s+1} - \dfrac{1}{(s^2+1)(s^2+s+1)} \end{pmatrix}$$

$$= \frac{\begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}}{s^2 + 1} + \begin{pmatrix} 1 & 0 \\ 0 & \frac{s+1}{s^2 + s + 1} \end{pmatrix}$$

$$= P + Q$$

$$\Gamma(P) = \begin{pmatrix} s + \frac{1}{s} & \frac{1}{s} \\ \frac{1}{s} & \frac{1}{s} \end{pmatrix} = Z''$$

$$\Gamma(Q) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{s + \frac{1}{s+1}} \end{pmatrix} = Z'$$

Hence, Z is realized by the network of Figure 3.

Example 2. Consider a problem given in [6, p. 275].

$$\hat{Z}(s) = \begin{pmatrix} \frac{2s^2 + 4s + 5}{2s^2 + s + 1} & \frac{s^2 + 2s + 2}{2s^2 + s + 1} \\ \frac{s^2 + 2s + 2}{2s^2 + s + 1} & \frac{s^2 + s + 1}{2s^2 + s + 1} \end{pmatrix}$$

$$\mathrm{Ev}\ \hat{Z} = \frac{\begin{pmatrix} 4s^4 + 8s^2 + 5 & 2s^4 + 3s^2 + 2 \\ & \\ 2s^4 + 3s^2 + 2 & 2s^4 + 2s^2 + 1 \end{pmatrix}}{4s^4 + 3s^2 + 1}$$

and

$$\det \mathrm{Ev}\ \hat{Z} = \frac{(s^2+1)^2\,(2s^2+1)^2}{(4s^4+3s^2+1)^2} \ .$$

Thus det Ev $\hat{Z}$ is already zero at $\pm j$, $\pm j\sqrt{2}/2$.

Choosing $\omega_o = 1$ we have

$$\mathrm{Re}\ \hat{Z}(j) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ & \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Thus, using

$$A = \begin{pmatrix} -1 & 1 \\ & \\ 1 & 0 \end{pmatrix}$$

we have $\mathrm{Re}(A_T\hat{Z}(j)A)_{11} = 0$. Hence

$$\tilde{Z} = A_T\hat{Z}A = \frac{\begin{pmatrix} s^2+s+2 & -(s^2+2s+3) \\ -(s^2+2s+3) & 2s^2+4s+5 \end{pmatrix}}{2s^2+s+1}$$

and at $s = j$, $\tilde{z}_{11} = -j$.

Thus

$$Z = \tilde{Z} + \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} 2(s^2+1)(s+1) & -(s^2+2s+3) \\ -(s^2+2s+3) & 2s^2+4s+5 \end{pmatrix}}{2s^2+s+1}$$

has a j-axis zero in z_{11}.

Forming $\Gamma(Z)$, we obtain

$$\Gamma(Z) = \frac{\begin{pmatrix} 2s^2 + s + 1 & s^2 + 2s + 3 \\ -(s^2 + 2s + 3) & \dfrac{-(s^2 + 2s + 3)^2}{2s^2 + s + 1} \end{pmatrix}}{2(s^2 + 1)(s + 1)}$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{2s^2 + 4s + 5}{2s^2 + s + 1} \end{pmatrix}$$

Since z_{11} was zero at $s = j$, $\Gamma(Z)$ has a pole at $s = j$.

The pole matrix is

$$P = \frac{\begin{pmatrix} s & 2 \\ -2 & 4s \end{pmatrix}}{2(s^2 + 1)}$$

which is of degree 2. P can be removed from $\Gamma(Z)$ to give

$$Z'' = \Gamma(P) = \begin{pmatrix} 2s + \frac{2}{s} & -\frac{2}{s} \\ -\frac{2}{s} & \frac{2}{s} \end{pmatrix}$$

Note that the degree of Z" is also 2.

Removing P from $\Gamma(Z)$ leaves $Q = \Gamma(Z) - P$. Then

$$Z' = \Gamma(Q) = \begin{pmatrix} 2s + 2 & -1 \\ -1 & \frac{4s^2 + s + 2}{4s^2 + 2s + 2} \end{pmatrix}$$

Since we added s to port 1 of $\tilde{Z}$, we must now subtract it. Also, to get $\hat{Z}$ we require the inverse congruence transformer at the input. Note that Z' and Z" posses series inductors in ports 1, which can be combined with -s as shown in Figure 4.

The final realization of $\hat{Z}(s)$ is given in Figure 5.

Note: The Brune transformer could be regarded as an ideal transformer and one inductor. Thus four

reactive elements are required, which is in accord with the fact that $\delta\hat{Z} = 4$.

The dots on the transformers have the following significance. When progressing along the winding from the dotted terminal, each winding encircles the core in the same sense. Thus, if dots are at the same ends of the windings, the coupling term in the matrix is positive; if at opposite ends, negative.

Proof of Theorems 11 and 12. Before commencing with the proof of Theorem 11, it is necessary to make some comments concerning notation. While the operator Γ was completely defined by Equations 4, it will be convenient in what foll ws to employ the following approach.

Let Z be partitioned into

$$\left(\begin{array}{c:c} z_{11} & \overrightarrow{z_{12}} \\ \hdashline z_{21}\!\downarrow & z_{22} \end{array}\right)$$

where

z_{11} is a scalar,

$\vec{Z}_{12}$ is a row-vector of n - 1 elements,

$Z_{21}\downarrow$ is a column-vector of n - 1 elements,

and

Z_{22} is an n - 1 x n - 1 matrix.

Regarding two n-vectors as n x 1 matrices, we have

$$X\downarrow \vec{Y} = \begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 & \cdots & x_1y_n \\ x_2y_1 & \cdots & & & x_2y_n \\ \vdots & & & & \\ x_ny_1 & x_ny_2 & x_ny_3 & \cdots & x_ny_n \end{pmatrix}$$

which is a rank 1, n x n matrix. Letting $\vec{\phi}$ be a null row-vector and ϕ a null matrix it can be verified that

$$\Gamma(Z) = (z_{11})^{-1} \left(\begin{array}{c} 1 \\ \hline Z_{21}\downarrow \end{array} \right) (1 \;\vdots\; -\vec{Z}_{12}) + \left(\begin{array}{c|c} 0 & \vec{\phi} \\ \hline \phi\downarrow & Z_{22} \end{array} \right)$$

is consistent with the definition of $\Gamma(Z)$ given in

Equation 4.

Proof of Theorem 11.

Proposition 1.

Form $\Gamma(Z)$. Since z_{11} has discrete zeros at $\pm j\omega_o$, $\Gamma(Z)$ has poles there. By Theorem 2 or the corollary to Theorem 7, $\Gamma(Z)$ is hybrid PR. Hence by Theorem 5, $\Gamma(Z)$ can be decomposed into a hybrid IPR matrix P, formed by the removal of the poles at $\pm j\omega_o$ from $\Gamma(Z)$ and a matrix $Q = \Gamma(Z) - P$, which is hybrid PR without j-axis poles.

It is shown in the proof of Proposition 3 below that p_{11} is not identically zero. Hence $Z'' = \Gamma(P)$ exists.

Suppose q_{11} is identically zero. Since p_{11} is IPR, it follows that $\mathrm{Re}(p_{11} + q_{11}) = 0$ everywhere on the j-axis. Hence by the corollary to Theorem 9, $z_{11} = \Gamma(p_{11} + q_{11})$ is also IPR. But this contradicts assumption c) of the statement of this theorem. Hence q_{11} is not identically zero and so $Z' = \Gamma(Q)$ exists.

P and Q share no common poles. Hence it follows by Equation 1 that $\delta(P + Q) = \delta P + \delta Q$. (For a detailed proof, see Theorem 7 of [5]).

But by Theorem 1, the degree is invariant under a gyration, so

$$\delta Z = \delta P + \delta Q = \delta Z' + \delta Z'' .$$

(It will be shown below that δP is 1 or 2). Hence $\delta Z' = \delta Z - \delta Z''$. Moreover Z' is symmetric PR since Q is hybrid PR. This completes the proof of Proposition 1.

<u>Propositions 2 and 3</u>.

By the PR property II, Re Z is non-negative definite for Re $s \geq 0$. Hence, $(\mathrm{Re}\ z_{11})(\mathrm{Re}\ z_{kk}) - (\mathrm{Re}\ z_{1k})^2 \geq 0$ on $s = j\omega$ for $k = 2, \ldots n$. But at $j\omega_o$, Re $z_{11} = 0$. Hence $z_{1k}(j\omega_o)$ is pure imaginary and we write at $s = j\omega_o$,

(7) $$\vec{Z}_{12} = j\,\vec{\alpha} = (Z_{21}|)_T .$$

Next define

(8) $$U = (z_{11})^{-1} \begin{pmatrix} 1 \\ \hline Z_{21}| \end{pmatrix} (1 \;\vdots\; -\vec{Z}_{12}) .$$

<u>Assume first that</u> $\omega_o \neq$ o <u>or</u> ∞. Then the hybrid IPR pole matrix removable from $\Gamma(Z)$ is given by

$$(9)\quad P = \frac{(s - j\omega_o)U\Big|_{s = j\omega_o}}{s - j\omega_o} + \frac{(s + j\omega_o)U\Big|_{s = -j\omega_o}}{s + j\omega_o}$$

$$= \frac{V}{s - j\omega_o} + \frac{V^*}{s + j\omega_o} \text{ say.}$$

Now by Equation 8, U is of Rank 1. Hence the residue matrices V and V* are likewise of Rank 1, and so, by Equation 1, P is of degree 2.

Let $V = M + jN$. Then

$$P = \frac{2sM - 2\omega_o N}{s^2 + \omega_o^2}$$

$$= \frac{2s}{s^2 + \omega_o^2}\left(\begin{array}{c|c} m_{11} & \vec{M}_{12} \\ \hline M_{21}\downarrow & M_{22} \end{array}\right) - \frac{2\omega_o}{s^2 + \omega_o^2}\left(\begin{array}{c|c} n_{11} & \vec{N}_{12} \\ \hline N_{21}\downarrow & N_{22} \end{array}\right)$$

Writing $V = \left(\begin{array}{c|c} v_{11} & \vec{v}_{12} \\ \hline v_{21}\downarrow & v_{22} \end{array}\right)$ we have by Equation 9,

$$v_{11} = \left.\frac{s - j\omega_o}{z_{11}}\right|_{s = j\omega_o} = b$$

say, which is real and positive since z_{11} is PR.

$$\vec{v}_{12} = -(s - j\omega_o) \left.\frac{\vec{z}_{12}}{z_{11}}\right|_{s = j\omega_o} = -jb\vec{\alpha} \quad \text{by Equation 7}$$

$$= -j\vec{\beta} \quad \text{say.}$$

$$v_{21}\downarrow = (s - j\omega_o) \left.\frac{z_{21}\downarrow}{z_{11}}\right|_{s = j\omega_o} = jb\alpha\downarrow \quad \text{by Equation 7}$$

$$= j\beta\downarrow \; .$$

Finally

$$v_{22} = -(s - j\omega_o) \left.\frac{(z_{21}\downarrow)\vec{z}_{12}}{z_{11}}\right|_{s = j\omega_o}$$

$$= b\alpha\downarrow\vec{\alpha}.$$

Hence

$$(10)\qquad P = \frac{2s}{s^2 + \omega_o^2}\left(\begin{array}{c|c} b & \vec{\phi} \\ \hline \phi\downarrow & b\alpha\downarrow\vec{\alpha} \end{array}\right) + \frac{2\omega_o}{s^2 + \omega_o^2}\left(\begin{array}{c|c} 0 & \vec{\beta} \\ \hline -\beta\downarrow & \phi \end{array}\right)$$

Define $Z'' = \Gamma(P)$, then

$$Z'' = \left(\begin{array}{c|c} \dfrac{s^2 + \omega_o^2}{2sb} & \dfrac{-\omega_o\vec{\beta}}{sb} \\ \hline \dfrac{-\omega_o\beta\downarrow}{sb} & \dfrac{2\beta\downarrow\vec{\beta}}{sb} \end{array}\right)$$

Z'' can be realized by Figure 6.

When $\omega_o = 0$ or ∞, the above analysis leads to Z'' being a matrix of degree 1 and consisting solely of a shunt reactor across port 1, with short circuits at the remaining ports.

This completes the proof of Theorem 11.

Proof of Theorem 12.

First we require the following:

Lemma 4. Let $\omega_o \neq 0$ or ∞. Then the matrix Z'' of Theorem 11 can be realized either by a series inductor and an ideal capacitive transformer (as proved previously) or by a series capacitor and an ideal inductive transformer.

Proof: Observe that for $\vec{\theta}$ a real constant vector,

$$\text{both } G = \left(\begin{array}{c|c} 0 & \vec{\theta} \\ \hline -\theta\!\downarrow & \Phi \end{array}\right)$$

and $-G$ are hybrid PR by Definition I, II, and III[1]. It is worth noting that they are realizable with one transformer and one gyrator.

We recall that in the proof of Theorem 11, the matrix $\Gamma(Z)$ was split into P and Q. The situation is not changed if we realize instead $P + G$ and $Q - G$. Both are hybrid PR and hence realizable. Moreover, $\delta(P + G) = \delta P$ and $\delta(Q - G) = \delta Q$ by Lemma 2.

Now, if we let $\vec{\theta} = 2\vec{\beta}/\omega_o$, then using P defined in Equation 10,

$$P + G = \left(\begin{array}{c|c} \dfrac{2sb}{s^2 + \omega_o^2} & \dfrac{-2s^2\,\vec{\beta}}{(s^2 + \omega_o^2)\omega_o} \\ \hline \dfrac{2s^2\beta'}{(s^2 + \omega_o^2)\omega_o} & \dfrac{2s\beta'\vec{\beta}}{(s^2 + \omega_o^2)b} \end{array}\right)$$

and

$$\Gamma(P + G) = \left(\begin{array}{c|c} \dfrac{s^2 + \omega_o^2}{2sb} & \dfrac{s}{\omega_o}\,\dfrac{\vec{\beta}}{b} \\ \hline \dfrac{s}{\omega_o}\,\dfrac{\beta'}{b} & \dfrac{2s}{\omega_o^2}\,\dfrac{\beta'\vec{\beta}}{b} \end{array}\right)$$

which is realized in Figure 7. This proves the lemma.

<u>We proceed with the proof of Theorem 12</u>. (A scalar version is found in [12]).

Consider first the case where for $s = j\omega_o$, then

$$\tilde{z}_{11}(j\omega_o) = -j\gamma\omega_o \qquad (\gamma > 0)\ .$$

Then we form Z with elements

$$z_{ij} = \tilde{z}_{ij} + \gamma s \delta_{i1}\delta_{j1},$$

where δ means the Kronecker delta.

But $z_{11} = \tilde{z}_{11} + \gamma s$ has a zero at $\pm j\omega_o$ and a pole at ∞. The pole matrix P has a one-one element $2sb/(s^2 + \omega_o^2)$, which was removed from $1/(\tilde{z}_{11} + \gamma s)$, leaving as the one-one element of Q the PR function:

$$f(s) = \frac{s^2 + \omega_o^2 - 2\gamma bs^2 - 2bs\tilde{z}_{11}}{(s^2 + \omega_o^2)(\tilde{z}_{11} + \gamma s)}$$

Recall that $\tilde{z}_{11}(s)$ is PR and has no pole at ∞.

The inverse of f(s) is the one-one element of Z' which yields a series pole of value $s\gamma/(1 - 2\gamma b)$. Note that $\gamma/(1 - 2\gamma b)$ is real and positive since it is the residue at a j-axis pole removed from a PR function. The first realization of this cycle is given in Figure 8.

The element $-\gamma s$ compensates for the γs added to $\tilde{z}_{11}$. The above realization can be given the equivalent network shown in Figure 9.

In case $\tilde{z}_{11}(j\omega_o) = j(\gamma/\omega_o)$ we add γ/s. Then taking account of Lemma 4 above, the final

realization of the Brune cycle becomes that of Figure 10.

The upper part of the diagram shows a "capacitive transformer". A two-port capacitive transformer is equivalent to a capacitor and an ideal transformer (see [6, p. 114]).

This completes the proof of Theorem 12.

CHAPTER III

Generalizing the Gyration

3.1 INTRODUCTION. In this chapter the definition and properties of the gyration are extended. It is shown that sets of gyrations can be specified which form Abelian groups of order 2^n. The generalized gyration provides an extended basis for the description of network parameters.

3.2 EXTENSION OF THE Γ OPERATOR

Suppose $Y\!\downarrow\, = A\, X\!\downarrow$ where A is an $n \times n$ matrix, $X\!\downarrow$ and $Y\!\downarrow$ are n-vectors.

Partitioning as follows

$$\begin{pmatrix} Y_1\!\downarrow \\ \hline Y_2\!\downarrow \end{pmatrix} = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \begin{pmatrix} X_1\!\downarrow \\ \hline X_2\!\downarrow \end{pmatrix}$$

where A is $n \times n$, A_{11} is $r \times r$, $X_1\!\downarrow$ and $Y_1\!\downarrow$ are r-vectors, we find by solving for $X_1\!\downarrow$ that if A_{11}^{-1} exists:

$$\begin{pmatrix} X_1\rangle \\ \hline Y_2\rangle \end{pmatrix} = \left\{ \begin{pmatrix} I_r \\ \hline A_{21} \end{pmatrix} A_{11}^{-1} \left(I_r \;\vdots\; -A_{12} \right) + \left(\begin{array}{c:c} \phi & \phi \\ \hdashline \phi & A_{22} \end{array} \right) \right\} \begin{pmatrix} Y_1\rangle \\ \hline X_2\rangle \end{pmatrix}$$

This interchange of $X_1\rangle$ and $Y_1\rangle$ is precisely the Γ operator when applied simultaneously to the first r pivots of A. We say then that

$$(11)\quad \Gamma_{1,2,\ldots r}(A) = \begin{pmatrix} I_r \\ \hline A_{21} \end{pmatrix} A_{11}^{-1} \left(I_r \;\vdots\; -A_{12} \right) + \left(\begin{array}{c:c} \phi & \phi \\ \hdashline \phi & A_{22} \end{array} \right)$$

defined whenever A_{11}^{-1} exists. $\Gamma_{1,2,\ldots r}(A)$ is termed an <u>r-fold gyration of A</u>.

Suppose we wish to perform an r-fold gyration on an arbitrary set of diagonal members of A, not necessarily the first r.

If $Y\rangle = A\,X\rangle$ and M is a permutation matrix, then

$$M_T Y\rangle = M_T A\, M\, M_T X\rangle$$

Let $M_T Y\downarrow = Y'\downarrow$, $M_T X\downarrow = X'\downarrow$, and $M_T A\ M = A'$. Then $Y'\downarrow = A'X'\downarrow$ will be a rearrangement of the equation $Y\downarrow = AX\downarrow$, in which any r pivots can be made to be the first r by an appropriate choice of M. The r-fold gyration on the selected set can now be performed according to the definition given in Equation (11) which is thus completely general.

Without loss of generality let r = 2, n = 3. Then from the standpoint of the vectors $X\downarrow$ and $Y\downarrow$, Equation (11) is equivalent to:

$$\Gamma_{1,2}(x_1\ x_2\ x_3\ y_1\ y_2\ y_3) = (y_1\ y_2\ x_3\ x_1\ x_2\ y_3)$$

and it is thus clear that

(12) $$\Gamma_{1,2}(A) = \Gamma_1[\Gamma_2(A)] = \Gamma_2[\Gamma_1(A)] .$$

Of course for r = 1, Equation (11) is consistent with the definition of Γ_1 given in Equations (4).

It follows from Equation (12) that Theorems 1 and 2 are true for the r-fold gyration operation.

Suppose F is a PR matrix and let G be an r-fold gyration of F. Then Theorem 3 generalizes

to the following:

Theorem 13. Conditions I and II are necessary and sufficient that G be the r-fold gyration of the impedance matrix of a passive network, provided that in the partitioning of G, the $r \times r$ submatrix G_{11} is not identically singular.

Theorem 4 can likewise be generalized as follows.

Theorem 14. Conditions I, II and IV are necessary and sufficient that G be the r-fold gyration of the impedance matrix of a network without resistors provided that in the partitioning of G, G_{11} is not identically singular.

It is important to note that in the above generalizations and in the generalizations to follow all symmetry conditions have been completely relaxed. The theorems will thus be fully applicable to PR matrices in general, and will not be restricted to the symmetric case.

3.3 REMOVAL OF IMAGINARY AXIS POLES FROM NON-RECIPROCAL MATRICES

Suppose F is a PR matrix, not necessarily symmetric. If f_{11} or det $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ is zero at

some discrete point on the j-axis then it is clear that the gyration matrices $\Gamma_1(F)$ or $\Gamma_{1,2}(F)$ will have poles on the j-axis. Let these gyration matrices be called G. Then by reasoning as in section 1.5, we have the following generalization.

<u>Theorem 15</u>. <u>Let</u> G <u>be the</u> r-fold <u>gyration of a</u> PR <u>matrix</u>. <u>Then</u>

$$G = P + Q$$

<u>where</u> P <u>is</u> IPR <u>and</u> Q <u>is</u> PR <u>without poles on the</u> j-axis.

3.4 GROUP PROPERTIES

For the case n = 3, let

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Subject to an appropriate gyration, each of the pairs $(y_1\ x_1)$, $(y_2\ x_2)$, $(y_3\ x_3)$ can be made to lie in the state $(y_i\ x_i)$ as in the above equation or in the gyrated state $(x_i\ y_i)$. Thus there are precisely 2^3 or 8 distinct gyrations which can be

defined for a 3 x 3 matrix. In case of an n x n matrix there are 2^n. The existence of all of the 2^n gyration matrices depends on the principal submatrices of the matrix A. The existence of a given gyration matrix follows only when a corresponding principal submatrix of A is non-singular.

Definition. The operation of following one gyration by a second is called a cascade.

$\Gamma_m[\Gamma_n(A)]$ generates a cascaded pair of operators $\Gamma_m \Gamma_n$.

Theorem 16. Let A be an n x n matrix for which all of the 2^n possible distinct gyration matrices exist. Then the gyration operators form an involutary Abelian group of order 2^n over the cascade operation.

Proof.

$$\Gamma_i[\Gamma_j(A)] = \Gamma_{ij}(A)$$

and so the set is closed over the cascade operation.

Let i j k be integers which lie between 0 and n. Define Γ_o as the gyration which interchanges none of the $(y_i\ x_i)$. Then

$\Gamma_i(\Gamma_j[\Gamma_k(A)]) = \Gamma_i[\Gamma_{jk}(A)] = \Gamma_{ij}[\Gamma_k(A)]$,

and so the set is associative over the cascade operation.

Let m be any sequence of integers between 0 and n, without repeating.

Then $\Gamma_o[\Gamma_m(A)] = \Gamma_m(A)$ and so Γ_o is the identity element of the set.

Since $\Gamma_m[\Gamma_m(A)] = A$ it follows that every element is its own inverse and so the set forms a group with the involutory property.

Finally $\Gamma_m[\Gamma_n(A)] = \Gamma_n[\Gamma_m(A)]$ and so the group is Abelian.

This completes the proof.

The group can be symbolically displayed by the use of a hypercube of dimension n.

Let n = 3; then the cube is 3-dimensional and can be thought of as lying in the first hyper-octant of an orthogonal triplet of axes a,b,c. (See Figure 34).

The gyration matrices can be associated with the corners of the cube as follows.

Suppose m is an ordered string of integers without repeats chosen from 1,2,3 or n = 0.

Then $\Gamma_m(A)$ is a gyration matrix which is one of possibly 8 distinct matrices. If m includes i, then the i-th pair $(y_i\ x_i)$ is interchanged and so the i-th coordinate is given the value 1 in the a,b,c space. The following table shows the relationship between $\Gamma_m(A)$ and its coordinates in the a,b,c system.

TABLE I

GYRATION MATRIX	COORDINATES		
	a	b	c
$\Gamma_o(A)$	0	0	0
$\Gamma_1(A)$	1	0	0
$\Gamma_2(A)$	0	1	0
$\Gamma_3(A)$	0	0	1
$\Gamma_{1,2}(A)$	1	1	0
$\Gamma_{1,3}(A)$	1	0	1
$\Gamma_{2,3}(A)$	0	1	1
$\Gamma_{1,2,3}(A)$	1	1	1

Figure 34 shows the location of $\Gamma_m(A)$ and its coordinates.

The operators themselves can be associated with spatial directions in the a,b,c space.

Γ_1 will change any matrix from its given form to a new form along an edge parallel to the a-axis.

Γ_2 operates parallel to the b-axis and Γ_3 parallel to the c-axis.

As an example $\Gamma_3(A)$ has position coordinates (0,0,1). The Γ_1 operation gives

$$\Gamma_1[\Gamma_3(A)] = \Gamma_{1,3}(A)$$

which has coordinates (1,0,1) and is given by a move along the "a" direction.

Crossing the diagonal of a cube face is associated with a 2-fold gyration. There are three 2-fold gyrations associated with the three planar senses of the cube faces. Crossing a major diagonal of the cube is associated with the 3-fold gyration. The 3-fold gyration is equivalent to matrix inversion in this case, and so each of the gyration matrices is faced by its inverse at the opposite end of its corresponding major diagonal.

Starting with a given matrix A, we may or may not be able to perform all of the 2^n gyrations

possible, depending on the existence of A_{11}^{-1}. Thus the group may or may not be complete.

Theorem 17. For a given matrix A the group of operators is complete if and only if every principal submatrix of A is non-singular.

Proof. Suppose the group of operators is complete. Then it is possible starting from A, to move either along an edge, across the diagonal of a face or across a major diagonal of the cube and thereby to arrive at any other corner of the cube.

Hence, every principal submatrix of A must be non-singular.

Conversly suppose every principal submatrix of A is non-singular. Then by Equation 11 every one of the 2^n operators is defined.

This completes the proof.

3.5 APPLICATION TO NETWORK THEORY

Suppose a passive network gives rise to an impedance matrix Z(s). Then Z is PR. Let it be of degree d and suppose that at $s = s_o$ the rank of its hermitian part is r. If all of the principal minors of $Z(s_o)$ are non-zero, then from Theorem 17, every gyration matrix is defined at s_o. It

follows that a complete hypercube can be drawn whose corners are associated with $Z(s_o)$, its inverse $Y(s_o)$ and all of the gyration matrices $\Gamma_m[Z(s_o)]$. Each of these matrices is PR, of degree d and has a hermitian part of rank m, a result which follows from Theorems 1, 2 and the corollary to Theorem 9.

If any of the principal submatrices are singular at s_o, but not identically so, then the appropriate gyration matrix has a pole at s_o. (It is this property which underlies all of the synthesis theory developed in Chapters 2 and 4).

If any of the principal submatrices are identically singular, then that gyration matrix is not defined and the cube is incomplete.

Consider the following examples.

<u>Example 1</u>

A scalar $z = \frac{s^2 + 1}{s}$.

The hypercube is 1-dimensional and has two corners, one at the orgin and one at the point 1, together with the joining edge. It is complete. (See Figure 35)

At the origin is z with poles at s = 0 and s = ∞, zeroes at $\pm j$ and a hermitian part which is zero on the j-axis. z is PR and of degree 2.

At the point 1 lies $y = z^{-1}$ which has poles at $\pm j$, zeros at 0 and ∞. y has a zero hermitian part on the j-axis, is PR and of degree 2.

Example 2.

$$Z = s\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Z is IPR, degree 1, and the 2 x 2 matrix Z is identically singular but its 1 x 1 principal submatrices are non-singular, although zero at s = 0.

The 2-dimensional hypercube is thus degenerate and is shown in Figure 36. It contains only two edges and the three points (0,0), (1,0) and (0,1). At (0,0) lies Z. At (1,0) lies

$$\Gamma_1(Z) = \begin{pmatrix} \frac{1}{s} & -1 \\ 1 & 0 \end{pmatrix}$$

which has a pole at $s = 0$, is IPR and of degree 1.

At (0,1) lies

$$\Gamma_2(Z) = \begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{s} \end{pmatrix}$$

with the same properties as Z.

$Z^{-1} = \Gamma_{1,2}(Z)$ is not defined.

Example 3.

A unity turns ratio, ideal transformer is certainly PR but has no impedance or admittance matrix. It is defined instead by the hybrid equation

$$\begin{pmatrix} v_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ v_2 \end{pmatrix}$$

The inverse of the hybrid matrix exists, but its 1 x 1 principal submatrices are both singular and so a 1-fold gyration is never possible. Hence only two corners and the connecting diagonal of the 2-dimensional hypercube exist. The degenerate cube is depicted in

Figure 37 . The impedance and admittance matrices are undefined. Both existing corners are associated with an IPR matrix of degree 0 with a null Hermitian part.

Example 4.

Consider the null matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, IPR of degree 0.

If this is regarded as an impedance matrix then it is realizable by two short circuits. No gyrations are possible and so the two-dimensional cube degenerates into a single point at (0,0).

As an admittance matrix it is realizable by two open circuits, and has a degenerate cube consisting of a single point at (1,1).

As a hybrid matrix $\Gamma_1(Z)$ it is realizable as a short across port 2 and an open circuit in port 1. This follows from the fact that in this case

$$\begin{pmatrix} i_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ i_2 \end{pmatrix} = \Gamma_1(Z) \begin{pmatrix} v_1 \\ i_2 \end{pmatrix}$$

As the hybrid $\Gamma_2(Z)$ it is likewise realized by a short and open circuit. In the last two cases as in the first two the cube degenerates into the

single points (1,0) and (0,1) respectively.

CHAPTER IV

General PR Immittance Matrix Synthesis

4.1 INTRODUCTION In this final chapter the gyration operator is used as the basis for a general synthesis procedure for PR immittance matrices. The synthesis procedure derived in Chapter II is a special case of this general procedure. In this chapter the symmetry condition is not however, required and both non-symmetric and symmetric matrices can be realized by this general procedure.

4.2 COMMENTS ON THE DEFINITION OF A PR MATRIX

Let F(s) be a matrix function of the complex variable s all of whose elements f_{ij} are rational with real coefficients.

F can be split into the sum of two matrices

$$F = F_H + F_{SH}$$

where F_H is hermitian and F_{SH} is skew-hermitian. This follows immediately from the identity

$$F = \frac{F + F_T^*}{2} + \frac{F - F_T^*}{2}$$

Letting

$$F_H = \frac{F + F_T^*}{2}$$

we see that $F_H = F_{H_T}^*$ and is thus hermitian.

Similarly if

$$F_{SH} = \frac{F - F_T^*}{2}$$

then $F_{SH} = - F_{SH_T}^*$ and is thus skew-hermitian.

Condition II of the definition of the PR property for matrices requires that Re $\overline{x}^* F x| \geq 0$ for Re $s \geq 0$ and arbitrary $\overline{x}$. Now

$$\overline{x}^* F x| = \overline{x}^* F_H x| + \overline{x}^* F_{SH} x|$$

$$= \alpha + j\beta$$

where α and $j\beta$ are real and pure imaginary since they are hermitian and skew-hermitian forms respectively. Hence II may be restated as

II^1 The hermitian part of F is non-negative definite for Re s $\geq$ 0.

IIa, IIb and IIc can be restated as

II_a^1 F_H is non-negative definite for Re s = 0

II_b^1 f_{ij} has no poles for Re s > 0
$i,j = 1,2, \dots n.$

II_c^1 For Re s = 0, f_{ij} has simple poles and the matrix of residues is hermitian non-negative definite.

Likewise IV is equivalent to:

IV^1 A positive real matrix F is said to be IPR if on the line Re s = 0, F is everywhere skew-hermitian.

Thus the rank of the hermitian part of an IPR matrix is zero everywhere on the j-axis.

As will be seen later, this restatement of the PR property in matrix terms rather than the scalar terms previously used will be advantageous under certain conditions.

4.3 SYNTHESIS PROCEDURE

In what follows we develop a synthesis procedure, using the Γ operator, which is capable of handling any PR immittance matrix. As will be shown, any such matrix falls into precisely one of two classes and accordingly the procedure will have two variants, termed <u>Case A</u> and <u>Case B</u>. Case A will be shown to be a generalization of the procedure already given in Chapter II for symmetric matrices and will handle all symmetric PR as well as some non-symmetric PR matrices. Case B will cover all remaining PR matrices.

Suppose we are given a PR impedance matrix $\hat{Z}$ which is to be realized. As a first step, all imaginary axis poles are removed by splitting the matrix into the sum of pole matrices and a remainder which has no j-axis poles. All of these matrices are PR and the pole matrices being IPR, can be readily synthesized by inspection using inductors, capacitors and possibly gyrators together with ideal transformers.

Assume now a PR impedance matrix $\hat{Z}$ without j-axis poles. As discussed in section 2.3, by

removal of resistance from each port or from just one port a situation is reached where the hermitian part $\hat{Z}_H$ becomes singular at some point on the j-axis say $j\omega_o$. In Lemma 5 below it is shown that there exists a congruence transformation which, at $s = j\omega_o$, places a zero in the 1,1 position of $\hat{Z}_H$ or else places a singular 2 x 2 submatrix in the first principal position of $\hat{Z}_H$.

Lemma 5. Let F be an n x n matrix which is not identically singular and has no j-axis poles. Let F be PR and let $F_H(j\omega_o)$ be singular.

Then there exists a real, constant nonsingular matrix D such that either

i) $D_T F_H(j\omega_o) D$ has a zero in the 1,1 position (Case A).

or

ii) $D_T F_H(j\omega_o) D$ has a 2 x 2 submatrix in the first principal position which is singular (Case B).

Proof. $F_H(j\omega_o)$ is singular by assumption. Two cases arise. Either Re $F_H(j\omega_o)$ is singular (Case A) or it is not (Case B).

<u>Case A</u> F_H and Re F_H are singular at $j\omega_o$. There exists D, real and non-singular such that $D_T \text{Re } F_H(j\omega_o)D$ is diagonal and has a zero in the 1,1 position. But $F_H = \text{Re } F_H + j \text{ Im } F_H$ and Im F_H is skew-symmetric, a property which is preserved under congruence.

Hence $D_T F_H(j\omega_o)D$ has a zero in the 1,1 position. <u>This proves Case A.</u>

<u>Case B</u> Re $F_H(j\omega_o)$ is non-singular. By the PR property of F, $F_H(j\omega_o)$ is non-negative definite (Condition II[1], section 4.2). Hence

$$\vec{x}^* F_H(j\omega_o) x\rangle = \vec{x}^* \text{Re } F_H(j\omega_o) x\rangle + j\vec{x}^* \text{Im } F_H(j\omega_o) x\rangle \geq 0$$

for all $\vec{x}$.

In particular, for all real $\vec{x}$ since Im $F_H(j\omega_o)$ is skew-symmetric,

$$\vec{x} \text{Im } F_H(j\omega_o) x\rangle = 0$$

giving

$$\vec{x} \text{Re } F_H(j\omega_o) x\rangle \geq 0$$

Hence Re $F_H(j\omega_o)$ is non-negative definite and since Re $F_H(j\omega_o)$ is non-singular by assumption, it is in fact positive definite. Thus there exists

a matrix Q which is real and non-singular such that

$$Q_T \text{Re } F_H(j\omega_o)Q = I$$

Let

$$Q_T \text{Im } F_H(j\omega_o)Q = N, \text{ say}$$

where N is real skew-symmetric. It is shown in Appendix 2 that there exists an orthogonal matrix V such that if the real skew-symmetric matrix N is of even order then $V_T N\ V =$

$$\begin{pmatrix} \begin{array}{cc|cc} 0 & \mu_1 & & \\ -\mu_1 & 0 & & \\ \hline & & 0 & \mu_2 \\ & & -\mu_2 & 0 \end{array} & & 0 \\ & \ddots & \\ 0 & & \begin{array}{|cc} \hline 0 & \mu_n \\ -\mu_n & 0 \end{array} \end{pmatrix}$$

and if N is of odd order, then $V_T N\ V =$

$$\begin{pmatrix} \begin{array}{cc|cc|c} 0 & \mu_1 & & & \\ -\mu_1 & 0 & & & \\ \hline & & 0 & \mu_2 & \\ & & -\mu_2 & 0 & 0 \\ \hline & 0 & & & \ddots \end{array} & & \\ & \begin{array}{|cc|} \hline 0 & \mu_n \\ -\mu_n & 0 \\ \hline \end{array} & \\ & & 0 \end{pmatrix}$$

In either case some of the μ_i may be zero. Taking $D = QV$, we obtain $D_T F_H(j\omega_o)D =$

$$\begin{pmatrix} \begin{array}{cc|cc} 1 & j\mu_1 & & \\ -j\mu_1 & 1 & & \\ \hline & & 1 & j\mu_2 \\ & & -j\mu_2 & 1 \end{array} & \\ & \ddots & \\ & & \begin{array}{|cc} \hline 1 & j\mu_n \\ -j\mu_n & 1 \end{array} \end{pmatrix}$$

(assuming that n is even), which has ones in every diagonal position.

But $F_H(j\omega_o)$ is singular. Hence for some k,

$$\begin{pmatrix} 1 & j\mu_k \\ -j\mu_k & 1 \end{pmatrix}$$

is singular. Without loss of generality, let this be true for k = 1. Then $D_T F_H(j\omega_o) D =$

$$\left(\begin{array}{cc:cc:c:cc} 1 & j & & & & & \\ -j & 1 & & & & & \\ \hdashline & & 1 & j\mu_2 & & & \\ & & -j\mu_2 & 1 & & 0 & \\ \hdashline & & & & \ddots & & \\ \hdashline & 0 & & & & 1 & j\mu_n \\ & & & & & -j\mu_n & 1 \end{array}\right)$$

<u>which proves Case B, and completes the proof of Lemma 5.</u>

Thus after $\hat{Z}_H$ has been made singular at $j\omega_o$ by the removal of resistance the matrix is found to fall into Class A (all those for which Re $\hat{Z}_H$ is also singular at $j\omega_o$) or Class B (those for which Re $\hat{Z}_H$ is positive definite). We observe that all symmetric matrices immediately fall into Class A since for them the hermitian part is real.

We now apply the appropriate congruence transformation, as given by Lemma 5 and we can thus assume without loss of generality that

$$D_T \hat{Z} D = \tilde{Z}$$

has either a hermitian part whose 1,1 element is zero or else it has a hermitian part whose first principal 2 x 2 submatrix is singular at $j\omega_o$.

This congruence transformation by D, can be compensated for by also applying the inverse congruence transformation, the latter being realized by the appropriate use of ideal transformers as discussed in [6].

Thus far, the synthesis procedure has resulted in the network of Figure 12

4.4 CASE A SYNTHESIS

Assume $\tilde{Z}$ to be an n x n PR impedance matrix, without j-axis poles, with $\tilde{z}_{H_{11}}(j\omega_o) = 0$.

If $\tilde{z}_{11}$ is identically zero, we have the following

Lemma 6. Let F be an n x n PR matrix with $f_{11} = 0$ identically. Then

$$F = \left(\begin{array}{c|c} f_{11} & \vec{F}_{12} \\ \hline F_{21}\downarrow & F_{22} \end{array}\right) = \left(\begin{array}{c|c} 0 & \vec{\alpha} \\ \hline -\alpha\downarrow & F_{22} \end{array}\right)$$

where $\vec{\alpha}$ is a real constant row-vector, and F_{22} is PR.

Proof. F_{22} is PR since it is a principal submatrix of a PR matrix. F_H is hermitian non-negative definite for Re s ≥ 0. Since $f_{11} = 0$, we must have

$$\vec{F}_{12} + (F_{21}\downarrow)^*_T = \vec{\phi} \text{ for Re } s \geq 0.$$

Let s be real and positive. Then $(F_{21}\downarrow)^* = F_{21}\downarrow$.
Hence $\vec{F}_{12} + (F_{21}\downarrow)_T = \vec{\phi}$ for $s > 0$, and so $\vec{F}_{12} = -(F_{21}\downarrow)_T$ for all s.

We then have, $\vec{F}_{12} - \vec{F}^*_{12} = \vec{\phi}$ for Re s ≥ 0.

Let s be on the j-axis.

Then $\vec{F}_{12} - \vec{F}_{12}^* = \vec{\phi}$ implies that $\vec{F}_{12}$ is real on the entire j-axis.

Hence $\vec{F}_{12} = \alpha$ a real constant n-1 vector.

<u>This completes the proof of the lemma</u>.

If $\tilde{z}_{11}$ is identically zero, we may by Lemma 6, split $\tilde{Z}$ into

$$\left(\begin{array}{c|c} 0 & \vec{\alpha} \\ \hline -\alpha' & \phi \end{array}\right) + \left(\begin{array}{c|c} 0 & \vec{\phi} \\ \hline \phi' & \tilde{Z}_{22} \end{array}\right)$$

The first of these two matrices can be realized by a gyrator and ideal transformers (see later) and we then resume synthesis on $\tilde{Z}_{22}$. Thus we may assume without loss of generality that $\tilde{z}_{11}$ is not identically zero.

Suppose next that $\tilde{z}_{H_{11}}$ is identically zero on the j-axis. Then $\tilde{z}_{11}$ is IPR, and since it cannot have poles on thej-axis, (these have all been removed), $\tilde{z}_{11}$ must be identically zero. But this has already been ruled out. Thus we may assume that $\tilde{z}_{H_{11}}$ is not identically zero on the j-axis.

Suppose finally that $\tilde{Z}$ or one of its principal minors is identically singular. We then have

Lemma 7. Let F(s) be a PR matrix which is identically singular. Then F is congruent to

$$\left(\begin{array}{c|c} 0 & \vec{\alpha} \\ \hline -\alpha\downarrow & W \end{array}\right)$$

where W is a PR matrix and $\vec{\alpha}$ is a real, constant row-vector.

Proof. Take s = 1. Let F(1) = M + N where M is real symmetric and N real skew-symmetric. Let C be a real constant, non-singular matrix.

Then $C_T F(1) C = C_T M\ C$.

But $C_T F(1) C$ is singular. Hence so is M. Choose C so that

$$C_T M\ C = \left(\begin{array}{c|c} 0 & \vec{\phi} \\ \hline \phi\downarrow & M' \end{array}\right)$$

Then $[C_T F(s) C]_{11}$ has a zero in Re $s > 0$. But this implies that $[C_T F(s) C]_{11} = 0$ identically since it is PR.

Hence by Lemma 6

$$C_T F(s) C = \left(\begin{array}{c|c} 0 & \vec{\alpha} \\ \hline -\alpha\downarrow & W \end{array}\right)$$

where W is PR and $\vec{\alpha}$ is a real and constant row vector. <u>This completes the proof of the lemma</u>.

<u>Corollary</u>. <u>If</u> F <u>is</u> PR <u>and identically singular</u>, <u>then</u> F_H <u>is identically singular</u>.

<u>Proof</u>. There exists a non-singular C such that

$$C_T F\ C = \left(\begin{array}{c|c} 0 & \vec{\alpha} \\ \hline -\alpha\downarrow & W \end{array}\right)$$

Hence

$$F_H = C_T^{-1} \left(\begin{array}{c|c} 0 & \vec{\phi} \\ \hline \phi\downarrow & W_H \end{array}\right) C^{-1}$$

<u>which proves the assertion</u>.

Thus we may assume that none of the principal submatrices of $\tilde{Z}$ is identically singular, since if this were the case, we can apply a congruence transformation, remove a gyrator section and resume synthesis of the remainder.

Without loss of generality, then, $\tilde{Z}$ is an $n \times n$ PR impedance matrix, without j-axis poles, no principal submatrices identically singular, $\tilde{z}_{H_{11}}$ is not identically zero on the j-axis and $\tilde{z}_{H_{11}} = 0$ at $j\omega_o$.

By Theorem 17, every one of the possible 2^n gyration matrices is defined for $\tilde{Z}$.

Following the classical Brune tradition [12] we now add a scalar $b = sL$ or $\frac{1}{sC}$ to $\tilde{z}_{11}$ so that $\tilde{z}_{11} + b$ is zero at $j\omega_0$. If $\omega_o = 0$ or ∞ then $\tilde{z}_{H_{11}}(j\omega_o) = 0$ implies $\tilde{z}_{11}(j\omega_o) = 0$ and so $b = 0$ in these two cases.

Define

$$Z = \tilde{Z} + \left(\begin{array}{c|c} b & \vec{\phi} \\ \hline \vec{\phi}^t & \phi \end{array}\right)$$

Then $z_{11}(j\omega_o) = 0$ but z_{11} is not identically zero.

$\tilde{Z}$ has no j-axis poles; it follows that Z has only the j-axis pole possibly due to b. If $0 < \omega_o < \infty$, then Z has at most one j-axis pole in only z_{11} and this pole is either at 0 or ∞, depending on b. On the other hand if $\omega_o = 0$ or ∞ then $b = 0$ and so Z then has no j-axis poles.

We are now in a position to reduce the degree of Z by the removal of a lossless section.

4.5 STATEMENT AND PROOF OF CASE A THEOREMS

Theorem 18. Let $\tilde{Z}(s)$ be an n x n PR impedance matrix,

a) with no identically singular principal submatrices,

b) without j-axis poles,

c) with $\tilde{z}_{H_{11}}$ not identically zero on the j-axis, and

d) $\tilde{z}_{H_{11}} = 0$ at $j\omega_o$.

Let $Z = \tilde{Z} + \left(\begin{array}{c|c} b & \bar{\phi} \\ \hline \phi & \phi \end{array}\right)$ where the IPR scalar b is so chosen that $z_{11}(j\omega_o) = 0$. Then Z may be decomposed in the series parallel manner of Figure 1 where

1. Z' is PR and $\delta Z' = \delta Z - \delta Z''$

2. Z" is IPR and $\delta Z''$ is 1 or 2

3. Z" contains at most 2 reactors and 1 gyrator plus ideal transformers.

Since z_{11} has zeros at $\pm j\omega_o$, $\Gamma_1(Z)$ has poles there. $\Gamma_1(Z)$ is PR. Hence by Theorem 15, $\Gamma_1(Z)$ can be split into an IPR matrix P, formed by the removal of the poles at $\pm j\omega_o$ from $\Gamma_1(Z)$, plus a matrix $Q = \Gamma_1(Z) - P$ which is PR without poles at $\pm j\omega_o$.

As will be shown under the proof of Proposition 3, p_{11} is not identically zero and so $Z'' = \Gamma_1(P)$ exists.

Suppose q_{11} is identically zero. Since p_{11} is IPR, it has a real part which is identically zero on the j-axis. It follows from the corollary to Theorem 9 that $z_{11} = \Gamma_1(p_{11} + q_{11})$ also has a real part which is identically zero on the j-axis.

But this contrary to assumption c in the statement of this theorem. Hence q_{11} cannot be identically zero and so $Z' = \Gamma_1(Q)$ exists.

By Theorem 2, both Z' and Z'' are PR. Since P and Q share no poles it follows from Equation 1, that $\delta(P + Q) = \delta P + \delta Q$. By Theorem 1, the degree is invariant under a gyration, so

$$\delta Z = \delta(P + Q) = \delta Z' + \delta Z''.$$

Hence $$\delta Z' = \delta Z - \delta Z''$$

We shall refer to these as propositions 1, 2, and 3 respectively.

This theorem is the basis of the Case A procedure. In order to compensate for the addition of b(s) we must also add -b(s) to $\tilde{z}_{11}$. It turns out, analagously to the scalar case, that this subsequent negative element can be incorporated into a perfectly coupled transformer. This is proved in

<u>Theorem 19</u>. <u>Let</u> Z, $\tilde{Z}$ <u>and</u> b <u>be as defined in</u> Theorem 18. <u>Then after the application of that theorem to split</u> Z, <u>it is always possible to incorporate the compensating negative reactor into a perfectly coupled transformer</u>.

<u>Proof of Theorem 18.</u>

<u>Proposition 1.</u>

$$\text{Form } \Gamma_1(Z) = (z_{11})^{-1}\begin{pmatrix} 1 \\ \hline \vec{z}_{21} \end{pmatrix}\left(1 \;\vdots\; -\vec{z}_{12}\right) + \left(\begin{array}{c|c} 0 & \vec{\phi} \\ \hline \vec{\phi} & Z_{22} \end{array}\right)$$

<u>This completes the proof of Proposition 1</u>. Prior to proceeding with the proofs of the remaining propositions we make the following observation.

Let G be a real skew-symmetric n x n matrix. Then both G and - G have a non-negative definite hermitian part and so they are PR. Thus P + G and Q - G are also PR and it is thus permissible to split $\Gamma_1(Z)$ into P + G and Q - G. Since $\delta G = 0$ it follows then that all the arguments given in the above proof of Proposition 1 are equally valid if $Z'' = \Gamma_1(P + G)$ and $Z' = \Gamma_1(Q - G)$.

G will be used to a definite advantage in the proof of Theorem 19 to follow.

<u>Proposition 2</u>.

$$\text{Let } U = (z_{11})^{-1} \begin{pmatrix} 1 \\ \hline \overrightarrow{z_{21}} \end{pmatrix} \left(1 \;\vdots\; -\overrightarrow{z_{12}}\right)$$

<u>First let $0 < \omega_o < \infty$</u>

Then the pole matrix removable from $\Gamma_1(Z)$ is

$$P = \frac{(s - j\omega_o)U\Big|_{s=j\omega_o}}{s - j\omega_o} + \frac{(s + j\omega_o)U\Big|_{s=-j\omega_o}}{s + j\omega_o}$$

$$= \frac{V}{s - j\omega_o} + \frac{V^*}{s + j\omega_o} \quad \text{say.}$$

Now U is of rank 1. Hence the residue matrices V and V* are of rank 1. Thus by Equation 1, P is of degree 2, which means that by Theorem 1, Z" = Γ_1(P) is of degree 2. P is IPR, and so the rank of the hermitian part of P is zero on the j-axis (Condition IV^1, section 4.2)

Thus by Theorem 2, Z" is PR and by the corollary to Theorem 9, Z" is IPR. Hence Z" is of degree 2 and IPR.

<u>Let ω_o = 0 or ∞</u>

Suppose for definiteness that ω_o = 0. (The case ω_o = ∞ can be analyzed as the dual).

Then $P = \dfrac{sU\Big|_{s=0}}{s}$. U is of rank 1, and so

P is of degree 1. P is IPR. Hence Z" = Γ_1(P) is of degree 1 and IPR.

Thus for $0 \leq \omega_o \leq \infty$, Z" is IPR and of degree 1 or 2.

This completes the proof of Proposition 2.

Before proving proposition 3, we prove

Lemma 8. Let F be partitioned as follows

$$F = \left(\begin{array}{c|c} f_{11} & \vec{F}_{12} \\ \hline F_{21}\downarrow & F_{22} \end{array}\right) \quad \text{where } f_{11} \text{ is a scalar.}$$

Let F be PR with $f_{H_{11}}(j\omega_o) = 0$, but not identically zero. Then

$$\vec{F}_{12}(j\omega_o) = \vec{\alpha} + j\vec{\beta}$$

$$F_{21}(j\omega_o)\downarrow = -\alpha\downarrow + j\beta\downarrow$$

where $\vec{\alpha}$ and $\vec{\beta}$ are real $(n - 1)$ vectors.

Proof. F_H is non-negative definite on $s = j\omega$ and since $f_{H_{11}}(j\omega_o) = 0$, we have at $j\omega_o$,

$$f_{H_{1k}} = 0, \quad f_{H_{k1}} = 0, \qquad k = 2, 3, \ldots n$$

Thus the entire first row and column of $F(j\omega_o)$ have skew-hermitian symmetry which proves the lemma.

Proposition 3

As before $U = (z_{11})^{-1} \begin{pmatrix} 1 \\ --- \\ z_{21} \end{pmatrix} (1 \mid -\vec{z}_{12})$

Assume $0 < \omega_o < \infty$

Then the pole matrix removable from $\Gamma_1(Z)$ is

$$P = \frac{(s - j\omega_o)U\Big|_{s=j\omega_o}}{s - j\omega_o} + \frac{(s + j\omega_o)U\Big|_{s=-j\omega_o}}{s + j\omega_o}$$

$$= \frac{V}{s - j\omega_o} + \frac{V^*}{s + j\omega_o}$$

Let $V = M + jN$

Then $P = \dfrac{2sM - 2\omega_o N}{s^2 + \omega_o^2}$

$$= \frac{2s}{s^2 + \omega_o^2}\left(\begin{array}{c|c} m_{11} & \overrightarrow{M_{12}} \\ \hline M_{21}\!\downarrow & M_{22} \end{array}\right) - \frac{2\omega_o}{s^2 + \omega_o^2}\left(\begin{array}{c|c} n_{11} & \overrightarrow{N_{12}} \\ \hline N_{21}\!\downarrow & N_{22} \end{array}\right)$$

Let $V = \left(\begin{array}{c|c} v_{11} & \overrightarrow{V_{12}} \\ \hline V_{21}\!\downarrow & V_{22} \end{array}\right)$

Then $v_{11} = (s - j\omega_o) \left.\frac{1}{z_{11}}\right|_{s=j\omega_o} = \gamma$ say,

which is real and positive since z_{11} is PR.

i.e. $m_{11} = \gamma$ and $n_{11} = 0$.

$$\overrightarrow{V_{12}} = -(s - j\omega_o) \left.\frac{\overrightarrow{z_{12}}}{z_{11}}\right|_{s=j\omega_o}$$

$$= -\gamma(\vec{\alpha} + j\vec{\beta}) \text{ by Lemma 8, and so}$$

$$\overrightarrow{M_{12}} = -\gamma\vec{\alpha},\ \overrightarrow{N_{12}} = -\gamma\vec{\beta}.$$

Similarly $V_{21}\!\downarrow = \gamma(-\alpha\!\downarrow + j\beta\!\downarrow)$ by Lemma 8.

$$= M_{21\downarrow} + jN_{21\downarrow}$$

and finally

$$V_{22} = -(s - j\omega_o)\,\frac{Z_{21\downarrow}\,\vec{Z}_{12}}{z_{11}}\Bigg|_{s=j\omega_o}$$

$$= -\gamma(-\alpha\!\downarrow + j\beta\!\downarrow)(\vec{\alpha} + j\vec{\beta})$$

$$= \gamma[(\alpha\!\downarrow\vec{\alpha} + \beta\!\downarrow\vec{\beta}) + j(\alpha\!\downarrow\vec{\beta} - \beta\!\downarrow\vec{\alpha})]$$

$$= M_{22} + jN_{22}\ .$$

$$\text{Hence } P = \frac{2s\gamma}{s^2 + \omega_o^2}\left(\begin{array}{c|c} 1 & -\vec{\alpha} \\ \hline -\alpha\!\downarrow & \alpha\!\downarrow\vec{\alpha} + \beta\!\downarrow\vec{\beta} \end{array}\right)$$

$$- \frac{2\omega_o\gamma}{s^2 + \omega_o^2}\left(\begin{array}{c|c} 0 & -\vec{\beta} \\ \hline \beta\!\downarrow & \alpha\!\downarrow\vec{\beta} - \beta\!\downarrow\vec{\alpha} \end{array}\right)$$

Then $Z'' = \Gamma_1(P)$

$$
= \left(\begin{array}{c:c}
\dfrac{s^2+\omega_o^2}{2s\gamma} & \vec{\alpha} - \dfrac{\omega_o}{s}\vec{\beta} \\ \hdashline
\begin{array}{c} -\alpha\!\downarrow \\ \\ -\dfrac{\omega_o}{s}\beta\!\downarrow \end{array} &
\begin{array}{l}
\dfrac{2s\gamma}{s^2+\omega_o^2}\left(-\dfrac{\omega_o}{s}\beta\!\downarrow\vec{\alpha} + \dfrac{\omega_o}{s^2}\beta\!\downarrow\vec{\beta}\right) \\
+\dfrac{2s\gamma}{s^2+\omega_o^2}\left(-\alpha\!\downarrow\vec{\alpha} + \dfrac{\omega_o}{s}\beta\!\downarrow\vec{\beta}\right) \\
+\dfrac{2s\gamma}{s^2+\omega_o^2}\left(\alpha\!\downarrow\vec{\alpha} + \beta\!\downarrow\vec{\beta}\right) \\
-\dfrac{2\omega_o\gamma}{s^2+\omega_o^2}\left(\alpha\!\downarrow\vec{\beta} - \beta\!\downarrow\vec{\alpha}\right)
\end{array}
\end{array}\right)
$$

$$
= \left(\begin{array}{c:c}
\dfrac{s}{2\gamma} & \vec{\phi} \\ \hdashline
\phi\!\downarrow & \phi
\end{array}\right)
+ \frac{1}{s}\left(\begin{array}{c:c}
\dfrac{\omega_o^2}{2\gamma} & -\omega_o\vec{\beta} \\ \hdashline
-\omega_o\beta\!\downarrow & 2\gamma\beta\!\downarrow\vec{\beta}
\end{array}\right)
$$

$$
+ \left(\begin{array}{c:c}
0 & \vec{\alpha} \\ \hdashline
-\alpha\!\downarrow & \phi
\end{array}\right)
$$

Thus Z" has been split into the three matrices given in the preceeding equation. The first of the three matrices of Z" may be realized as shown in Figure 13. The second term of Z" can be realized as shown in Figure 14.

The third part of Z" namely

$$\left(\begin{array}{c|c} 0 & \vec{\alpha} \\ \hline -\alpha\downarrow & \phi \end{array}\right)$$

can be realized by one gyrator and a congruence transformer. This follows from the fact that is skew symmetric, real, of rank 2. Hence there exists a non-singular matrix V (see Appendix 2) such that

$$\left(\begin{array}{c|c} 0 & \vec{\alpha} \\ \hline -\alpha\downarrow & \phi \end{array}\right) = V_T^{-1} \left(\begin{array}{cc|c} 0 & 1 & \\ & & \phi \\ -1 & 0 & \\ \hline & \phi & \phi \end{array}\right) V^{-1}$$

which is realized in Figure 15. The overall network for Z" is shown in Figure 16.

<u>An alternate realization for</u> Z" <u>is possible</u>. Let $\vec{\theta}$ be a real n-1 vector. Observe that both

$$G = \left(\begin{array}{c|c} 0 & -\vec{\theta} \\ \hline \theta| & \phi \end{array}\right)$$

and -G are real skew-symmetric and hence they are PR. Instead of splitting $\Gamma_1(Z)$ into P and Q we split it into P + G and Q - G as was anticipated in the comments after the proof of Proposition 1. Let

$$\vec{\theta} = \frac{2\gamma}{\omega_o}\vec{\beta}$$

Then

$$P + G = \frac{2s\gamma}{s^2+\omega_o^2}\left(\begin{array}{c|c} 1 & -\vec{\alpha} \\ \hline -\alpha| & \alpha|\vec{\alpha} + \beta|\vec{\beta} \end{array}\right)$$

$$+ \frac{2\gamma}{s^2+\omega_o^2}\left(\begin{array}{c|c} 0 & -\frac{s^2}{\omega_o}\vec{\beta} \\ \hline \frac{s^2}{\omega_o}\beta| & -\omega_o(\alpha|\vec{\beta} - \beta|\vec{\alpha}) \end{array}\right)$$

Hence $Z'' = \Gamma_1(P + G)$

$$
= \left(\begin{array}{c|c}
\dfrac{s^2 + \omega_o^2}{2s\gamma} & \vec{\alpha} + \dfrac{s}{\omega_o}\vec{\beta} \\
\hline
\begin{array}{c} -\alpha{\downarrow} \\ \\ + \dfrac{s}{\omega_o}\beta{\downarrow} \end{array} &
\begin{array}{l}
\dfrac{2s\gamma}{s^2 + \omega_o^2}\left(\begin{array}{l} -\alpha{\downarrow}\vec{\alpha} - \dfrac{s}{\omega_o}\alpha{\downarrow}\vec{\beta} \\ + \dfrac{s}{\omega_o}\beta{\downarrow}\vec{\alpha} + \dfrac{s^2}{\omega_o^2}\beta{\downarrow}\vec{\beta} \end{array}\right) \\
\\
+ \dfrac{2s\gamma}{s^2 + \omega_o^2}(\alpha{\downarrow}\vec{\alpha} + \beta{\downarrow}\vec{\beta}) \\
\\
- \dfrac{2\omega_o\gamma}{s^2 + \omega_o^2}(\alpha{\downarrow}\vec{\beta} - \beta{\downarrow}\vec{\alpha})
\end{array}
\end{array}\right)
$$

$$
= \left(\begin{array}{c|c}
\dfrac{\omega_o^2}{2s\gamma} & \vec{\phi} \\
\hline
\phi{\downarrow} & \phi
\end{array}\right)
+ s \left(\begin{array}{c|c}
\dfrac{1}{2\gamma} & \dfrac{1}{\omega_o}\vec{\beta} \\
\hline
\dfrac{1}{\omega_o}\beta{\downarrow} & \dfrac{2\gamma}{\omega_o^2}\beta{\downarrow}\vec{\beta}
\end{array}\right)
$$

$$
+ \left(\begin{array}{c|c}
0 & \vec{\alpha} \\
\hline
-\alpha{\downarrow} & \dfrac{2\gamma}{\omega_o}(\beta{\downarrow}\vec{\alpha} - \alpha{\downarrow}\vec{\beta})
\end{array}\right)
$$

Z" of the alternate realization, has been split into three matrices. The first of these terms is a capacitor in series with port 1 of the network of the remaining terms.

The second term is a single perfectly coupled transformer shown in Figure 17.

The third term can be realized by a single gyrator and a congruence transformer, since it is congruent to

$$\left(\begin{array}{c|c} 1 & \vec{\phi} \\ \hline \frac{2\gamma}{\omega_o}\beta\!\downarrow & I \end{array}\right)\left(\begin{array}{c|c} 0 & \vec{\alpha} \\ \hline -\alpha\!\downarrow & \phi \end{array}\right)\left(\begin{array}{c|c} 1 & \frac{2\gamma}{\omega_o}\vec{\beta} \\ \hline \phi\!\downarrow & I \end{array}\right)$$

The overall network for Z" is given in Figure 18.

<u>If $\omega_o = 0$ or ∞</u>

For definiteness assume $\omega_o = 0$ ($\omega_o = \infty$ can be analyzed as the dual).

Then $U = (z_{11})^{-1}\left(\begin{array}{c} 1 \\ \hline z_{21}\!\downarrow \end{array}\right)\left(\begin{array}{c|c} 1 & -\vec{z}_{12} \end{array}\right)$

The pole matrix $P = \frac{sU\Big|_{s=0}}{s} = \frac{V}{s}$ say.

$$v_{11} = \frac{s}{z_{11}}\Bigg|_{s=0} = \gamma \text{ which is real}$$

and positive.

$$\vec{V}_{12} = \frac{-s\vec{Z}_{12}}{z_{11}}\Bigg|_{s=0}$$

But $\vec{Z}_{12}(0)$ is real. Hence by Lemma 8, $\vec{Z}_{12}(0) = \vec{\alpha}$ and $Z_{21}(0)\!\downarrow = -\alpha\!\downarrow$. Thus

$$\vec{V}_{12} = -\gamma\vec{\alpha}$$

$$V_{21}\!\downarrow = -\gamma\alpha\!\downarrow$$

Finally

$$V_{22} = \gamma\alpha\!\downarrow\vec{\alpha}$$

$$\text{Hence } P = \frac{\gamma}{s}\left(\begin{array}{c|c} 1 & -\vec{\alpha} \\ \hline -\alpha\!\downarrow & \alpha\!\downarrow\vec{\alpha} \end{array}\right)$$

Then $Z'' = \Gamma_1(P)$

$$= \frac{s}{\gamma}\left(\begin{array}{c|c} 1 & \vec{\phi} \\ \hline \phi I & \phi \end{array}\right) + \left(\begin{array}{c|c} 0 & \vec{\alpha} \\ \hline -\alpha I & \phi \end{array}\right)$$

Thus Z'' can be realized by a single inductor, a single gyrator and a congruence transformer, as shown in Figure 19.

Hence Z'' contains at most two reactors, one gyrator and ideal transformers.

<u>This proves Proposition 3 and completes the proof of Theorem 18.</u>

<u>Proof of Theorem 19</u>.

We recall that $\tilde{Z}$ was partitioned as follows

$$\tilde{Z} = \left(\begin{array}{c|c} \tilde{z}_{11} & \tilde{Z}_{12} \\ \hline \tilde{Z}_{21} & \tilde{Z}_{22} \end{array}\right)$$

where $\tilde{z}_{11}$ is a scalar.

Then since $\tilde{z}_{H_{11}}(j\omega_o) = 0$, we have

$$\tilde{z}_{11}(j\omega_o) = j\eta$$

where η is a real constant.

<u>Let $0 < \omega_o < \infty$</u>

If $\eta > 0$ let $B = \frac{\omega_o \eta}{s} \left(\begin{array}{c|c} 1 & \bar{\phi} \\ \hline \phi\downarrow & \phi \end{array}\right)$

If $\eta < 0$ let $B = -\frac{s\eta}{\omega_o} \left(\begin{array}{c|c} 1 & \bar{\phi} \\ \hline \phi\downarrow & \phi \end{array}\right)$

If $\eta = 0$ let $B = \phi$.

<u>If $\omega_o = 0$ or ∞</u>

Then $\eta = 0$. Let $B = \phi$.

In each of the above cases the matrix B is PR.

Recall that $Z = \tilde{Z} + B$. Then Z is PR and

$z_{11}(j\omega_o) = 0$.

<u>Case 1</u> $z_{H_{11}}(j\omega_o) = j\eta \qquad \eta > 0$.

Then $z_{11} = \tilde{z}_{11} + \frac{\omega_o \eta}{s}$ has a zero at ω_o and a pole at the origin.

Recall that $p_{11} = \frac{2s\gamma}{s^2 + \omega_o^2}$, which is subtracted

from $\frac{1}{z_{11}}$ leaving

$$q_{11} = [\Gamma_1(Z) - P]_{11} = \frac{1}{\tilde{z}_{11} + \frac{\omega_o \eta}{s}} - \frac{2s\gamma}{s^2 + \omega_o^2}$$

$$= \frac{s^3 + s\omega_o^2 - 2s^2\gamma\tilde{z}_{11} - 2s\gamma\omega_o\eta}{(s^2 + \omega_o^2)(s\tilde{z}_{11} + \omega_o\eta)}$$

Hence $z_{11}' = \frac{1}{[\Gamma_1(Z) - P]_{11}}$ yields a pole at $s = 0$

of value $$\frac{\omega_o^2\eta}{s(\omega_o - 2\eta\gamma)}$$

Since z_{11}' is PR and $\eta > 0$, we have that

$$\frac{1}{\omega_o - 2\eta\gamma} > 0$$

Inverting p_{11} gives

$$z''_{11} = \frac{\omega_o^2}{2s\gamma} + \frac{s}{2\gamma}$$

and utilizing the alternate relization for Z" (Figure 18) we obtain the network of Figure 20 where the series capacitor $-\omega_o\eta / s$ is the compensation for adding

$$B = \frac{\omega_o\eta}{s}\left(\begin{array}{c|c} 1 & \vec{\phi} \\ \hline \phi\downarrow & \phi \end{array}\right)$$

to $\tilde{Z}$. Now the three capacitors of Figure 20 can be combined into a three-terminal network

$$T = \begin{pmatrix} \frac{\omega_o^2}{2s\gamma} - \frac{\omega_o\eta}{s} & \frac{\omega_o^2}{2s\gamma} \\ \frac{\omega_o^2}{2s\gamma} & \frac{\omega_o^2}{2s\gamma} + \frac{\omega_o^2\eta}{s(\omega_o - 2\eta\gamma)} \end{pmatrix}$$

$$= \frac{\omega_o^2}{2s\gamma} \begin{pmatrix} \frac{\omega_o - 2\gamma\eta}{\omega_o} & 1 \\ 1 & \frac{\omega_o}{\omega_o - 2\gamma\eta} \end{pmatrix}$$

T is PR since $\omega_o - 2\gamma\eta > 0$, and $0 < \omega_o < \infty$ Moreover T is of rank 1, and is therefore of degree 1. Thus T may be realized as shown in Figure 21 by a single capacitor plus an ideal transformer. This gives us the realization of Z shown in Figure 22.

Case 2.

$$\tilde{z}_{11}(j\omega_o) = j\eta \qquad \eta < 0$$

Then $z_{11} = \tilde{z}_{11} - \frac{s\eta}{\omega_o}$ has a zero at ω_o and a pole at ∞.

$q_{11} = [\Gamma_1(Z) - P]_{11}$. Hence,

$$z_{11}' = \frac{1}{[\Gamma_1(Z) - P]_{11}} = \left(\frac{1}{\tilde{z}_{11} - \frac{s\eta}{\omega_o}} - \frac{2s\gamma}{s^2 + \omega_o^2} \right)^{-1}$$

$$= \frac{(s^2 + \omega_o^2)(\omega_o\tilde{z}_{11} - s\eta)}{\omega_o s^2 + \omega_o^3 - 2s\gamma\tilde{z}_{11}\omega_o + 2s^2\gamma\eta}$$

which has a pole at ∞ given by

$$\frac{-s\eta}{\omega_o + 2\gamma\eta}$$

Since this term is Pr and $\eta < 0$, we have that $\omega_o + 2\gamma\eta > 0$. From Z" we obtai: the pole at ∞ of value $s/2\gamma$ (see the proof of Proposition 3 of Theorem 18). The three terminal matrix T is now:

$$T = \begin{pmatrix} \frac{s\eta}{\omega_o} + \frac{s}{2\gamma} & \frac{s}{2\gamma} \\ \frac{s}{2\gamma} & \frac{s}{2\gamma} - \frac{s\eta}{\omega_o + 2\gamma\eta} \end{pmatrix}$$

$$= \frac{s}{2\gamma} \begin{pmatrix} \frac{\omega_o + 2\gamma\eta}{\omega_o} & 1 \\ 1 & \frac{\omega_o}{\omega_o + 2\gamma\eta} \end{pmatrix}$$

which is PR since $\omega_o + 2\gamma\eta > 0$ and $0 < \omega_o < \infty$.

Again T is of degree 1 since it has rank 1. T is realized by the three terminal network of Figure 23 which is a single inductive element, and Z is realized by Figure 24. <u>This completes the proof of Theorem 19</u>.

Prior to commencing with a discussion of Case B, we make the following observations.

1) Referring to Figures 22 and 24, either

$$Z' - \frac{\omega_o^2 \eta}{s(\omega_o - 2\gamma\eta)} \left(\begin{array}{c|c} 1 & \vec{\Phi} \\ \hline \Phi\downarrow & \Phi \end{array}\right)$$

or

$$Z' + \frac{s\eta}{\omega_o + 2\gamma\eta} \left(\begin{array}{c|c} 1 & \vec{\Phi} \\ \hline \Phi\downarrow & \Phi \end{array}\right)$$

now remain to be synthesized. If $\omega_o = 0$ or ∞, then Z' remains. The cycle is now resumed by splitting off any j-axis poles followed by a removal of resistance until a new $\hat{Z}$ is produced for which $\hat{Z}_H$ is singular at some point on the j-axis. Either a Case A or a Case B cycle now commences.

2) We note that a buffer of degree 1 or 2 was removed from Z resulting in a degree reduction of 1 or 2. Thus the Case A cycle is minimal in the sense of Tellegen's Minimal Theorem stated in section 2.3. As will be shown later, the Case B cycle reduces the degree of Z by 2 with the removal of a buffer of degree 2 and so it is also minimal. Thus a given matrix of degree d will be synthesized, by this method, by a network containing precisely d reactive elements.

3) We note that at most 1 gyrator appears in the buffer for a Case A cycle, whether the degree of the buffer is 1 or 2. We will show that at most one gyrator is required per unit degree reduction.

4.6 CASE B SYNTHESIS

Assume that

$$\tilde{Z} = \left(\begin{array}{c|c} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \hline \tilde{Z}_{21} & \tilde{Z}_{22} \end{array}\right)$$

is an n x n PR impedance matrix. $\tilde{Z}_{11}$ is 2 x 2. Let $\tilde{Z}$ have no j-axis poles, and assume as in Case A that none of the principal submatrices of $\tilde{Z}$ is identically singular. Since $\tilde{Z}$ falls under Case B, then $\tilde{Z}_{H_{11}}(j\omega_o)$ is singular, but Re $\tilde{Z}_{H_{11}}(j\omega_o)$ is positive definite. This immediately means that $\omega_o \neq 0$ or ∞ since at those points $\tilde{Z}$ is real and so $\tilde{Z}_{H_{11}}(j\omega_o)$ is real and singular, which implies that a Case A synthesis is possible. Similarly $\tilde{Z}_{H_{11}}$ cannot be identically singular on the j-axis since then it is singular at $\omega_o = 0$ and ∞. A case A synthesis is then also possible.

Thus we may assume without loss of generality that for Case B, $\tilde{Z}$ is an n x n PR impedance matrix, without j-axis poles, with no identically singular principal submatrices, $\tilde{Z}_{H_{11}}$ not identically singular on the j-axis, $\tilde{Z}_{H_{11}}(j\omega_o)$ singular, Re $\tilde{Z}_{H_{11}}(j\omega_o)$ not singular and $0 < \omega_o < \infty$.

We now follow a procedure similar to the Brune method, and add to $\tilde{Z}_{11}$ a 2 x 2 matrix B(s) such that $\tilde{Z}_{11} + B$ is singular at $j\omega_o$. The existence of such a matrix is guaranteed by

Lemma 9. Let $\tilde{Z}_{11}$ be a 2 x 2 PR matrix with $\tilde{Z}_{11}(j\omega_o) = C + jD - j(E + jF)$ where C and E are real symmetric, D and F are real skew-symmetric and where C + jD is positive semi-definite (i.e. singular) but C is non-singular.

Then there exists a 2 x 2 PR matrix B(s) such that

i) $\tilde{Z}_{11}(j\omega_o) + B(j\omega_o)$ is singular

ii) B(s) can be realized by at most 1 inductor, 1 gyrator and 1 ideal transformer.

Proof. Let $B(j\omega_o) = j\lambda(C + jD) + j(E + jF)$ where λ is as yet unspecified. Then

$$\tilde{Z}_{11}(j\omega_o) + B(j\omega_o) = (1 + j\lambda)(C + jD)$$

which is singular as required by i) of the lemma. To give B(s) the properties required by ii) of the lemma, λ is chosen as follows. C is non-singular by assumption. Since $\tilde{Z}_{11}$ is PR, C is also non-negative definite. Hence C is in fact positive definite. Thus there exists a real, non-singular N such that

$$C = N_T N \qquad \text{and} \qquad E = N_T \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} N.$$

Then

$$\lambda C + E = N_T \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right\} N.$$

<u>Suppose that</u> $p > q$

Take $\lambda = -q$. Then $\lambda C + E = N_T \begin{pmatrix} p-q & 0 \\ C & 0 \end{pmatrix} N$

which is positive semi-definite.

Take $B(s) = \frac{s}{\omega_o} (\lambda C + E) - (\lambda D + F)$. Then

$$B(j\omega_o) = j\lambda(C + jD) + j(E + jF)$$

as required by i). Moreover $B(s)$ is realizable by a gyrator, a single reactor, (since $\lambda C + E$ is positive semi-definite) and ideal transformers as shown in Figure 25. Thus ii) is satisfied.

<u>Suppose</u> $p < q$

Take $\lambda = -p$ and follow the above argument. We

again obtain a B(s) which satisfies i) and ii).

Suppose $p = q$.

Take $\lambda = -p$. Then $\lambda C + E$ is a null matrix. Let

$$B(s) = -(\lambda D + F).$$

Hence $B(j\omega_o) = j\lambda(C + jD) + j(E + jF)$ as required and B(s) is realizable by a single gyrator.

Thus in every case B(s) exists which satisfies i) and ii) of the lemma.

This completes the proof.

We conclude from this lemma that B(s), a 2 x 2 PR matrix, can be found, whose addition makes $\tilde{Z}_{11}(j\omega_o)$ singular. This is an extension of the Brune procedure, and following the Brune tradition, B(s) will be compensated for at a later stage by the addition of -B(s), which, as will be shown, can be incorporated into a Pr network.

Let

$$Z = \tilde{Z} + \left(\begin{array}{c|c} B & \Phi \\ \hline \Phi & \Phi \end{array}\right)$$

Then $Z_{11}(j\omega_o)$ is singular but Z_{11} is not identically singular. Since $\tilde{Z}$ has no j-axis poles,

Z has only the j-axis poles contributed by B, and these occur only in Z_{11}. Since

$B = \frac{s}{\omega_o}(\lambda C + E) - (\lambda D + F)$, if $\lambda C + E$ is not the null matrix, Z_{11} has a rank 1 pole at ∞, and if $\lambda C + E$ is null, then Z_{11} has no poles. We are now in a position to reduce the degree of Z by the removal of a lossless section.

4.7 STATEMENT AND PROOFS OF CASE B THEOREMS

Theorem 20. Let $\tilde{Z}$ be an n x n PR impedance matrix

a) without j-axis poles

b) without identically singular principal submatrices

c) $\tilde{Z}_{H_{11}}$ not identically singular on the entire j-axis

d) $\tilde{Z}_{H_{11}}(j\omega_o)$ singular

e) Re $\tilde{Z}_{H_{11}}(j\omega_o)$ not singular and

f) $0 < \omega_o < \infty$.

Let $Z = \tilde{Z} + \left(\begin{array}{c|c} B & \phi \\ \hline \phi & \phi \end{array}\right)$ where B is IPR, 2 x 2 and so chosen by Lemma 9, that $Z_{11}(j\omega_o)$ is singular. Then Z may be decomposed in the manner of Figure 26 where

1) Z' is PR and $\delta Z' = \delta Z - 2$

2) Z'' is IPR and $\delta Z'' = 2$

3) Z'' consists of an inductive portion of degree 2, at most one gyrator plus ideal transformers.

We will refer to these as Propositions 1, 2, and 3 respectively.

This theorem is the basis of the Case B procedure. In order to compensate for the addition of B(s) we must also add -B(s) to $\tilde{Z}_{11}$. It turns out, analagously to the scalar case, that this subsequent negative matrix can be incorporated into a PR 6-terminal network. This is proved in Theorem 21. Given Z, $\tilde{Z}$ and B as defined in Theorem 20 and Lemma 9. Then, after the application of Theorem 20 to split Z, it is always possible to

<u>incorporate the compensating</u> $-B(s)$ <u>into a</u> PR <u>net-work</u>. Proof of Theorems 20 and 21 now follow.

<u>Proof of Theorem 20</u>.

<u>Proposition 1</u>

Form $$\Gamma_{1,2}(Z) = \begin{pmatrix} I_2 \\ \hline Z_{21} \end{pmatrix} Z_{11}^{-1} \left(I_2 \;\vdots\; -Z_{12} \right) + \left(\begin{array}{c|c} \phi & \phi \\ \hline \phi & Z_{22} \end{array} \right)$$

Since Z_{11} is singular at $\pm j\omega_o$, $\Gamma_{1,2}(Z)$ has poles there. By Theorem 2, $\Gamma_{1,2}(Z)$ is PR. Hence by Theorem 15, $\Gamma_{1,2}(Z)$ can be decomposed into an IPR matrix P, formed by the removal of the poles at $j\omega_o$ from $\Gamma_{1,2}(Z)$, and a matrix $Q = \Gamma_{1,2}(Z) - P$ which is PR without poles at $\pm j\omega_o$.

Partition P into $\left(\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array} \right)$ where P_{11}

is 2 x 2. Do likewise for Q. As is shown below in the proof of Proposition 3, P_{11} is not identically singular and so $Z" = \Gamma_{1,2}(P)$ exists. Q_{11} cannot be identically singular, for suppose it is. By the corollary to Lemma 7, $Q_{H_{11}}$ is then identically singular. Since P_{11} is IPR it follows that the hermitian part of $(P_{11} + Q_{11})$ is singular on the entire j-axis. Hence by the corollary to Theorem 9, $Z_{11} = \Gamma_{1,2}(P_{11} + Q_{11})$ has a hermitian part which is singular on the entire j-axis. This contradicts assumption c) of this theorem. Hence Q_{11} cannot be identically singular and so $Z = \Gamma_{1,2}(Q)$ exists.

By Theorem 2, both Z' and Z" are PR. P and Q share no common poles. Hence it follows by Equation 1 that

$$\delta(P + Q) = \delta P + \delta Q$$

But by Theorem 1, the degree is invariant under a gyration, so

$$\delta Z = \delta P + \delta Q = \delta Z' + \delta Z"$$

It will be shown below, in the proof of Proposition

3, that $\delta Z'' = 2$. Hence

$$\delta Z' = \delta Z - 2$$

<u>This completes the proof of Proposition 1</u>.

We make the following observation. Let G be a real skew-symmetric matrix. Then both G and - G are PR. If instead of splitting $\Gamma_{1,2}(Z)$ into P and Q, we split it into P + G and Q - G, then both P + G and Q - G are PR.

If we choose G so that it has the form

$$G = \left(\begin{array}{c|c} \Phi & \Phi \\ \hline \Phi & F \end{array}\right)$$

where F is $(n - 2) \times (n - 2)$ then by the same arguments as in the proof of Proposition 1, P_{11} and Q_{11} are not identically singular. Hence $Z'' = \Gamma_{1,2}(P + G)$ and $Z' = \Gamma_{1,2}(Q - G)$ will be defined, PR and of the same degrees as in Proposition 1. G will be useful in Proposition 3.

Prior to proving Proposition 2, we prove

<u>Lemma 11</u> Let M be a 2 x 2 matrix. Let A and B be 2 x r matrices. Then

$$N = \begin{pmatrix} I_2 \\ \hdashline A_T{}^* \end{pmatrix} M \, (I_2 \mid B) \text{ has the same rank as } M.$$

Moreover if $A = B$ and M is hermitian positive semi-definite, then N is hermitian positive semi-definite.

<u>Proof</u> Let $M = \begin{pmatrix} \vec{m}_1 \\ \vec{m}_2 \end{pmatrix}$ where $\vec{m}_1$ and $\vec{m}_2$ are 2-element row vectors. Then

$$\begin{pmatrix} I_2 \\ \hdashline A_T{}^* \end{pmatrix} M = \begin{pmatrix} \vec{m}_1 \\ \vec{m}_2 \\ \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_r \end{pmatrix} = K \text{ say where the } \vec{a}_i \text{ are linear}$$

combinations of $\vec{m}_1$ and $\vec{m}_2$. Let $K = (k_1\!\downarrow \; k_2\!\downarrow)$ where $k_1\!\downarrow$ and $k_2\!\downarrow$ are column vectors of order $r+2$. Then $K \, (I_2 \mid B) = (k_1\!\downarrow \; k_2\!\downarrow \; b_1\!\downarrow \; . \; . \; .b_r\!\downarrow) = N$ where the $b_i\!\downarrow$ are linear combinations of $k_1\!\downarrow$ and $k_2\!\downarrow$. Clearly the rank of N equals the rank of K and the rank of K equals the rank of M.

This proves the first part of the lemma. If A = B, let $Q = \left(\begin{array}{c|c} I_2 & -A \\ \hline \phi & I_r \end{array}\right)$. Then Q is non-singular.

Hence N is congruent to

$$Q_T{}^*N\,Q = \left(\begin{array}{c|c} M & \phi \\ \hline \phi & \phi \end{array}\right)$$

which completes the proof of the lemma.

Proof of Proposition 2

Recall that $0 < \omega_o < \infty$

Define $U = \left(\begin{array}{c} I_2 \\ \hline Z_{21} \end{array}\right) Z_{11}^{-1} \left(I_2 \;\vdots\; -Z_{12}\right)$

$Z_{11}(j\omega_o)$ is a singular 2 x 2 matrix. Hence its rank is zero or 1. If its rank is zero, then $z_{11}(j\omega_o) = 0$ and a Case A synthesis is possible. Thus we may assume that $Z_{11}(j\omega_o)$ is of rank 1. The residue matrix at $s = j\omega_o$ is

$$V = (s - j\omega_o)U\Big|_{s=j\omega_o}$$

$$= \frac{s - j\omega_o}{\det Z_{11}} \begin{pmatrix} I_2 \\ \hline Z_{21} \end{pmatrix} \operatorname{adj} Z_{11} \left(I_2 \;\vdots\; -Z_{12}\right)\Bigg|_{s=j\omega_o}$$

Since Z_{11} is 2 x 2 rank 1, adj Z_{11} has rank 1. It thus follows by Lemma 11 that V is of rank 1.

The pole matrix removable from $\Gamma_{1,2}(Z)$ is:

$$P = \frac{V}{s - j\omega_o} + \frac{V^*}{s + j\omega_o} \tag{13}$$

which is thus of degree 2. Moreover P is IPR by Theorem 15. It therefore follows, by Theorems 1 and 2 and the corollary to Theorem 9 that $Z'' = \Gamma_{1,2}(P)$ is IPR and of degree 2.

This proves Proposition 2.

<u>Proposition 3</u>.

The residue matrix at $s = j\omega_o$ in $\Gamma_{1,2}(Z)$ is

$$V = \frac{(s - j\omega_o)}{\det Z_{11}} \begin{pmatrix} I_2 \\ \hline Z_{21} \end{pmatrix} \operatorname{adj} Z_{11} \left(I_2 \;\vdots\; -Z_{12}\right)\Bigg|_{s=j\omega_o}$$

which is rank 1, as was noted above in the proof of Proposition 2. But V is the residue matrix at a j-axis pole of a PR matrix. Hence V is hermitian non-negative definite by condition II_c^1, (section 4.2). Partition V into

$$\left(\begin{array}{c:c} V_{11} & V_{12} \\ \hdashline V_{21} & V_{22} \end{array}\right)$$

where V_{11} is 2 x 2. Then V_{11} is also hermitian non-negative definite. Since

$$V_{11} = \frac{(s - j\omega_o)}{\det Z_{11}} \operatorname{adj} Z_{11} \bigg|_{s=j\omega_o}$$

and since Z_{11} is 2 x 2 rank 1, it follows that V_{11} is in fact hermitian positive semi-definite and must therefore be of the form

$$V_{11} = \begin{pmatrix} a_{11} & a_{12} + jb_{12} \\ \\ a_{12} - j_{12} & \dfrac{a_{12}^{\,2} + b_{12}^{\,2}}{a_{11}} \end{pmatrix}$$

where $a_{11} > 0$. Moreover $b_{12} \neq 0$ since if $b_{12} = 0$, then $Z_{11}(j\omega_o)$ is symmetric and singular and so a Case A synthesis is possible.

Having established the form which V_{11} must have we now determine the form of V. Let C and D be real 2 x n-2 matrices. If we take

$$V = \begin{pmatrix} I_2 \\ \hline C_T \\ -jD_T \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} + jb_{12} \\ a_{12} - jb_{12} & \dfrac{a_{12}^2 + b_{12}^2}{a_{11}} \end{pmatrix} (I_2 \mid C + jD)$$

then V is hermitian, and by Lemma 11, it is positive semi-definite, rank 1 as required. Let

$$W = \begin{pmatrix} a_{11} & a_{12} + jb_{12} \\ a_{12} - jb_{12} & \dfrac{a_{12}^2 + b_{12}^2}{a_{11}} \end{pmatrix}$$

Then expanding the above form for V we obtain

$$V = \left(\begin{array}{c|c} W & WC + jWD \\ \hline C_T W & C_T W\, C + D_T W\, D \\ -jD_T W & + jC_T W\, D - jD_T W\, C \end{array}\right)$$

Recall that $P(s) = \frac{V}{s - j\omega_o} + \frac{V^*}{s + j\omega_o}$

(see equation 13)

Let $V = M + jN$

Then $P = \frac{2sM - 2\omega_o N}{s^2 + \omega_o^2}$

Letting

$$K = \begin{pmatrix} sa_{11} & sa_{12} - \omega_o b_{12} \\ sa_{12} + \omega_o b_{12} & s\left(\frac{a_{12}^2 + b_{12}^2}{a_{11}}\right) \end{pmatrix}$$

and

$$L = \begin{pmatrix} \omega_o a_{11} & \omega_o a_{12} + sb_{12} \\ \omega_o a_{12} - sb_{12} & \omega_o\left(\frac{a_{12}^2 + b_{12}^2}{a_{11}}\right) \end{pmatrix}$$

we obtain

$$(14) \qquad P = \frac{2\begin{pmatrix} K & KC - LD \\ C_T K + D_T L & C_T K\, C + D_T K\, D - C_T L\, D + D_T L\, C \end{pmatrix}}{s^2 + \omega_o^2}$$

$$= \left(\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right)$$

We now add to P the matrix

$$\left(\begin{array}{c|c} \phi & \phi \\ \hline \phi & F \end{array}\right)$$

where F is real, skew-symmetric (n - 2) x (n - 2). As noted in the comments after the proof of Proposition 1, the addition of this matrix to P can be offset by subtracting it from Q. Then

$$Z'' = \left(\begin{array}{c} I_2 \\ \hline P_{21} \end{array}\right) P_{11}^{-1} \left(I_2 \mid -P_{12}\right) + \left(\begin{array}{c|c} \phi & \phi \\ \hline \phi & P_{22} + F \end{array}\right)$$

which yields

(15)

$$Z'' = \frac{s}{2b_{12}^{\,2}} \left(\begin{array}{cc|c} \dfrac{a_{12}^{\,2} + b_{12}^{\,2}}{a_{11}} & -a_{12} & \\ & & \phi \\ -a_{12} & a_{11} & \\ \hline & \phi & \phi \end{array}\right)$$

$$+ \left(\begin{array}{cc|c} 0 & \dfrac{\omega_o}{2b_{12}} & \\ & & M \\ \dfrac{-\omega_o}{2b_{12}} & 0 & \\ \hline & -M_T & F \end{array}\right)$$

where

$$M = C - \begin{pmatrix} -a_{12} & -\dfrac{a_{12}^{\,2} + b_{12}^{\,2}}{a_{11}} \\ & \\ a_{11} & a_{12} \end{pmatrix} \frac{D}{b_{12}}$$

is a real 2 x n-2 matrix.

The first part of Z" consists of a non-singular inductive matrix of degree 2 since $a_{11} > 0$ and $b_{12} \neq 0$. Hence this section of Z" can be realized by 2 inductors plus a congruence transformer. By the appropriate choice of F, the second part of Z" can be realized by 1 gyrator plus a congruence transformer. To see this, suppose first that $M = \begin{pmatrix} \vec{m}_1 \\ \vec{m}_2 \end{pmatrix}$ is of rank 2 and without loss of generality let $\frac{\omega_o}{2b_{12}} = 1$. Let $F = m_1\!\downarrow \vec{m}_2 - m_2\!\downarrow \vec{m}_1$. Observe that F is a skew symmetric real $(n - 2) \times (n - 2)$ matrix as required. Then the constant part of Z" is congruent to

$$H_T \left(\begin{array}{cc|c} 0 & 1 & \vec{m}_1 \\ -1 & 0 & \vec{m}_2 \\ \hline -m_1\!\downarrow & -m_2\!\downarrow & m_1\!\downarrow \vec{m}_2 - m_2\!\downarrow \vec{m}_1 \end{array}\right) H$$

$$= \left(\begin{array}{cc|c} 0 & 1 & \\ & & \phi \\ -1 & 0 & \\ \hline & \phi & \phi \end{array}\right)$$

where

$$H = \left(\begin{array}{cc|c} 1 & 0 & \vec{m}_2 \\ 0 & 1 & -\vec{m}_1 \\ \hline \multicolumn{2}{c|}{\Phi} & I \end{array}\right)$$

Thus the constant part of Z" can be realized by 1 gyrator and ideal transformers. On the other hand, if M is of rank 1, i.e. $M = \begin{pmatrix} \vec{m}_1 \\ \theta\vec{m}_1 \end{pmatrix}$ then letting F be the null matrix yields the result that only 1 gyrator plus a congruence transformer is required. <u>This completes Proposition 3 and Theorem 20</u>. The connections of Z' and Z" will be discussed in the proof of Theorem 21 which now follows.

<u>Proof of Theorem 21</u>

As required for a Case B synthesis $\tilde{Z}_{H_{11}}(j\omega_o)$ was a singular 2 x 2 matrix where $0 < \omega_o < \infty$.

We then added B(s), a 2 x 2 matrix, chosen according to Lemma 9, such that

$$B(s) = \frac{s}{\omega_o}(\lambda C + E) - (\lambda D + F)$$

has a real symmetric part $\lambda C + E$ of rank 1, a real skew-symmetric part $\lambda D + \nabla$ and has the property that $\tilde{Z}'_{11} + B$ is singular at $s = j\omega_o$.

Now
$$\left[\Gamma_{1,2}\left\{\tilde{Z} + \left(\begin{array}{c|c} B & \phi \\ \hline \phi & \phi \end{array}\right)\right\}\right]_{11} = (\tilde{Z}_{11} + B)^{-1}$$

Hence $Z'_{11} = [(\tilde{Z}_{11} + B)^{-1} - P_{11}]^{-1}$ where P is the pole matrix in $\Gamma_{1,2}\left\{\tilde{Z} + \left(\begin{array}{c|c} B & \phi \\ \hline \phi & \phi \end{array}\right)\right\}$ at $j\omega_o$ (see Equation 14). i.e.

$$Z'_{11} = (\tilde{Z}_{11} + B)[\, I - P_{11}(\tilde{Z}_{11} + B)\,]^{-1}$$

From Equation 14, we see that near $s = \infty$,
$P_{11} \doteqdot \frac{2A}{s}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & \dfrac{a_{12}^{\,2} + b_{12}^{\,2}}{a_{11}} \end{pmatrix}$$

is positive definite since $a_{11} > 0$ and $b_{12} \neq 0$. Moreover, $\tilde{Z}$ (and hence $\tilde{Z}_{11}$) contains no j-axis poles by assumption. Thus near $s = \infty$,

$$P_{11}(\tilde{Z}_{11} + B) \doteq \frac{2}{\omega_o} A(\lambda C + E)$$

and it follows then that Z'_{11} has a 2 x 2 pole at ∞ given by:

$$M = \frac{s}{\omega_o}(\lambda C + E)\left[I - \frac{2}{\omega_o} A(\lambda C + E)\right]^{-1}$$

which can be removed from Z' leaving behind a PR remainder.

Next, by Equation 15, Z" has a 2 x 2 pole matrix in its 1,1 2,2 position

$$N = \frac{s}{2b_{12}^2}\begin{pmatrix} \dfrac{a_{12}^2 + b_{12}^2}{a_{11}} & -a_{12} \\ -a_{12} & a_{11} \end{pmatrix}$$

$$= \frac{s}{2b_{12}^2} \quad A^{-1}$$

which can be removed from Z" leaving a PR remainder,

$$\left(\begin{array}{cc|c} 0 & \frac{\omega_o}{2b_{12}} & \\ & & K \\ \frac{-\omega_o}{2b_{12}} & 0 & \\ \hline & & \\ -K_T & & F \end{array}\right)$$

Thus the connections for the realization of Z are as shown in Figure 27.

The addition of -B is to compensate for the addition of B to Z. Observe the common terminals 1g and 2g. There are in fact three external terminals (1a, 1b, 1c) associated with port 1. Similarly for port 2. This gives a total of six terminals and two commons for the network made up of -B, N and M.

-B is not PR, but it can be absorbed into a 6-terminal (4 port) matrix given by

$$T = \left(\begin{array}{c|c} -B + N & N \\ \hline N & N + M \end{array}\right)$$

shown in Figure 28.

To see that T is PR we proceed as follows.

$$-B + N = [\frac{-s}{\omega_o}(\lambda C + E) + \lambda D + F] + \frac{s}{2} A^{-1}$$

$$= \frac{s}{2} A^{-1}[I - \frac{2}{\omega_o} A(\lambda C + E)] + \lambda D + F$$

Define $Q = I - \frac{2}{\omega_o} A(\lambda C + E)$

and $G = \lambda D + F$

Then $-B + N = \frac{s}{2} A^{-1}Q + G$

Also, $N + M = \frac{s}{2} A^{-1} + \frac{s}{\omega_o}(\lambda C + E)[I - \frac{2}{\omega_o} A(\lambda C + E)]^{-1}$

$$= \frac{s}{2} A^{-1}[I - \frac{2}{\omega_o} A(\lambda C + E)]^{-1}$$

$$= \frac{s}{2} A^{-1}Q^{-1}$$

Hence $T = \frac{s}{2}\left(\begin{array}{c|c} A^{-1}Q & A^{-1} \\ \hline A^{-1} & A^{-1}Q^{-1} \end{array}\right) + \left(\begin{array}{c|c} G & \phi \\ \hline \phi & \phi \end{array}\right)$

We note first that $\frac{s}{2} A^{-1}Q^{-1}$ is PR since it was formed from the addition of M and N, both PR.

$A^{-1}Q^{-1}$ being the residue at a j-axis pole, must be hermitian non-negative definite by condition III_c^1. By condition I, $\frac{s}{2} A^{-1}Q^{-1}$ must be real for s real. Hence $A^{-1}Q^{-1}$ is symmetric non-negative definite.

Since both Q and Q^{-1} exist, it follows that Q^{-1} is non-singular. A is positive definite.

Hence $A^{-1}Q^{-1}$ and its inverse QA are both symmetric positive-definite 2 x 2 matrices. We then have:

$$\frac{s}{2}\left(\begin{array}{c|c} A^{-1}Q & A^{-1} \\ \hline A^{-1} & A^{-1}Q^{-1} \end{array}\right)$$

$$= \frac{s}{2} R_T \left(\begin{array}{c|c} A^{-1/2}\ QA\ A^{-1/2} & I \\ \hline I & A^{1/2}\ A^{-1}Q^{-1}\ A^{1/2} \end{array}\right) R$$

where

$$R = \left(\begin{array}{c|c} A^{-1/2} & \phi \\ \hline \phi & A^{-1/2} \end{array}\right)$$

But
$$\left(\begin{array}{c|c} A^{-1/2}\,QA\,A^{-1/2} & I \\ \hline I & A^{1/2}\,A^{-1}Q^{-1}\,A^{1/2} \end{array}\right)$$

$$= X_T \left(\begin{array}{c|c} \phi & \phi \\ \hline \phi & A^{1/2}\,A^{-1}Q^{-1}\,A^{1/2} \end{array}\right) X$$

where
$$X = \left(\begin{array}{c|c} I & \phi \\ \hline A^{-1/2}\,QA\,A^{-1/2} & I \end{array}\right)$$

Hence the reactive part of T is congruent to

$$\frac{s}{2}\left(\begin{array}{c|c} \phi & \phi \\ \hline \phi & A^{-1}Q^{-1} \end{array}\right)$$

and can therefore be synthesized by two inductors and a congruence transformer. The constant part of T is 2 x 2, real skew-symmetric and can thus be synthesized by a single gyrator.

This completes the proof of Theorem 19.

We are now able to prove the following:

Theorem 22 Let Z be a PR matrix of degree d. Then the synthesis of Z, if carried out by the hybrid matrix method described in Theorems 18, 19, 20 and 21 will result in a lossless buffer containing d reactors, at most d gyrators and a termination of resistors and gyrators, plus ideal transformers.

Proof. Assume a PR matrix $\hat{Z}$ to be synthesized. The removal of an entire j-axis pole at any time, will reduce the degree by precisely the degree of the pole matrix, which appears in the buffer.

Suppose a Case A cycle is called for. Let $0 < \omega_o < \infty$. Let $\delta\tilde{Z} = m$. Hence $\delta Z = m + 1$ since $\delta\begin{pmatrix} b & \phi \\ \phi & \phi \end{pmatrix}$ of Theorem 18 is 1.

Now $\delta Z'' = 2$ and $\delta Z' = m - 1$. The absorption of $-b$ into the perfectly coupled transformer requires one reactor from Z'' and one from Z'. This transformer then has degree 1. Thus the buffer has degree 2, one due to the transformer and one remaining in Z''. The degree of the termination is $m - 2$. Thus the cycle is minimal.

Suppose $\omega_o = 0$ or ∞. Then no Brune impedance is added. $\delta Z'' = 1$ and $\delta Z' = m - 1$ and again the method is minimal.

Suppose a Case B cycle is called for. Let $\delta\tilde{Z} = m$. Then $\delta Z = \delta(\tilde{Z} + B) = m + 1$, since $\delta B = 1$ and B and $\tilde{Z}$ contain no common poles.

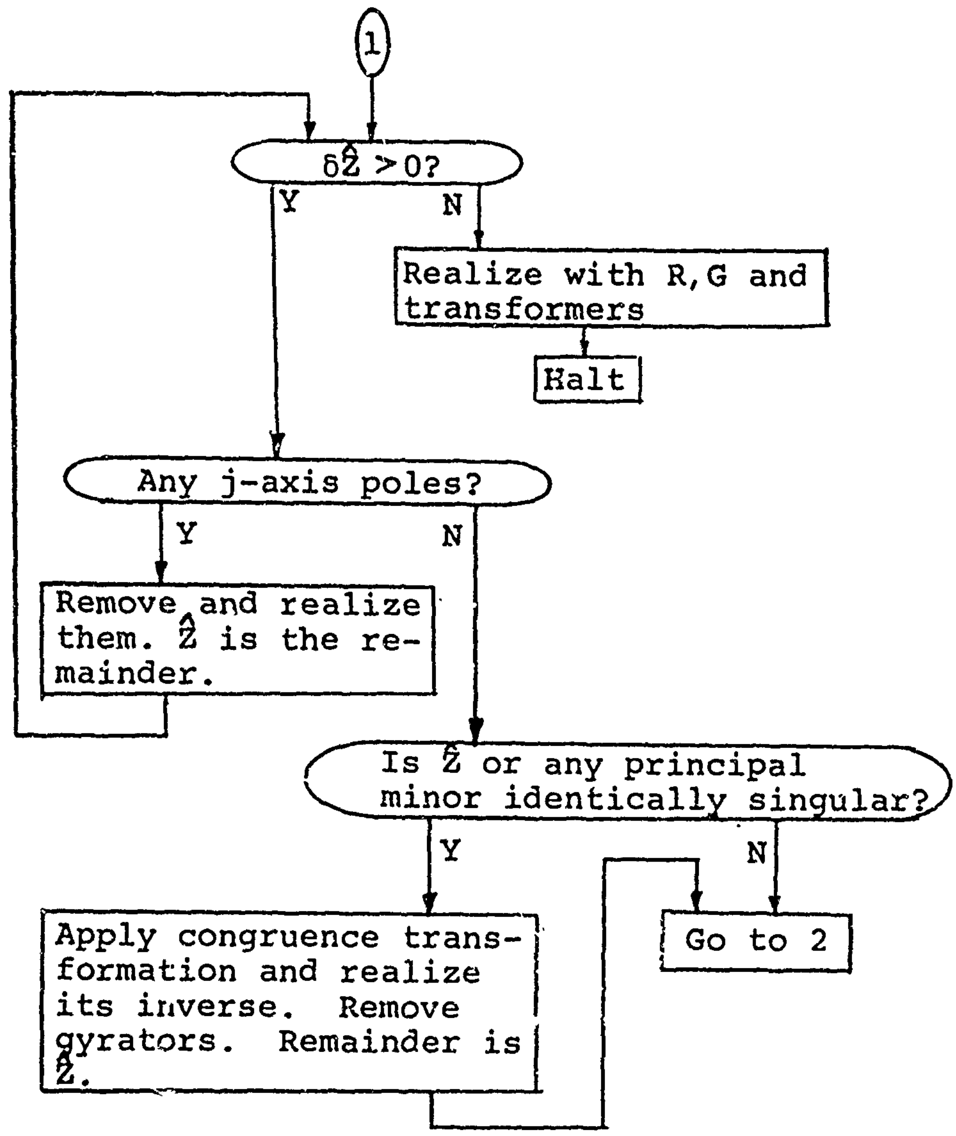
$\delta Z'' = 2$. Hence $\delta Z' = m - 1$. Both reactors in Z" are used to absorb -B, resulting in a 6 terminal reactive buffer of degree 2. Z' contributed a reactive matrix of degree 1 to the buffer. Thus the degree of the termination is $m - 2$. Thus in every case the method is minimal.

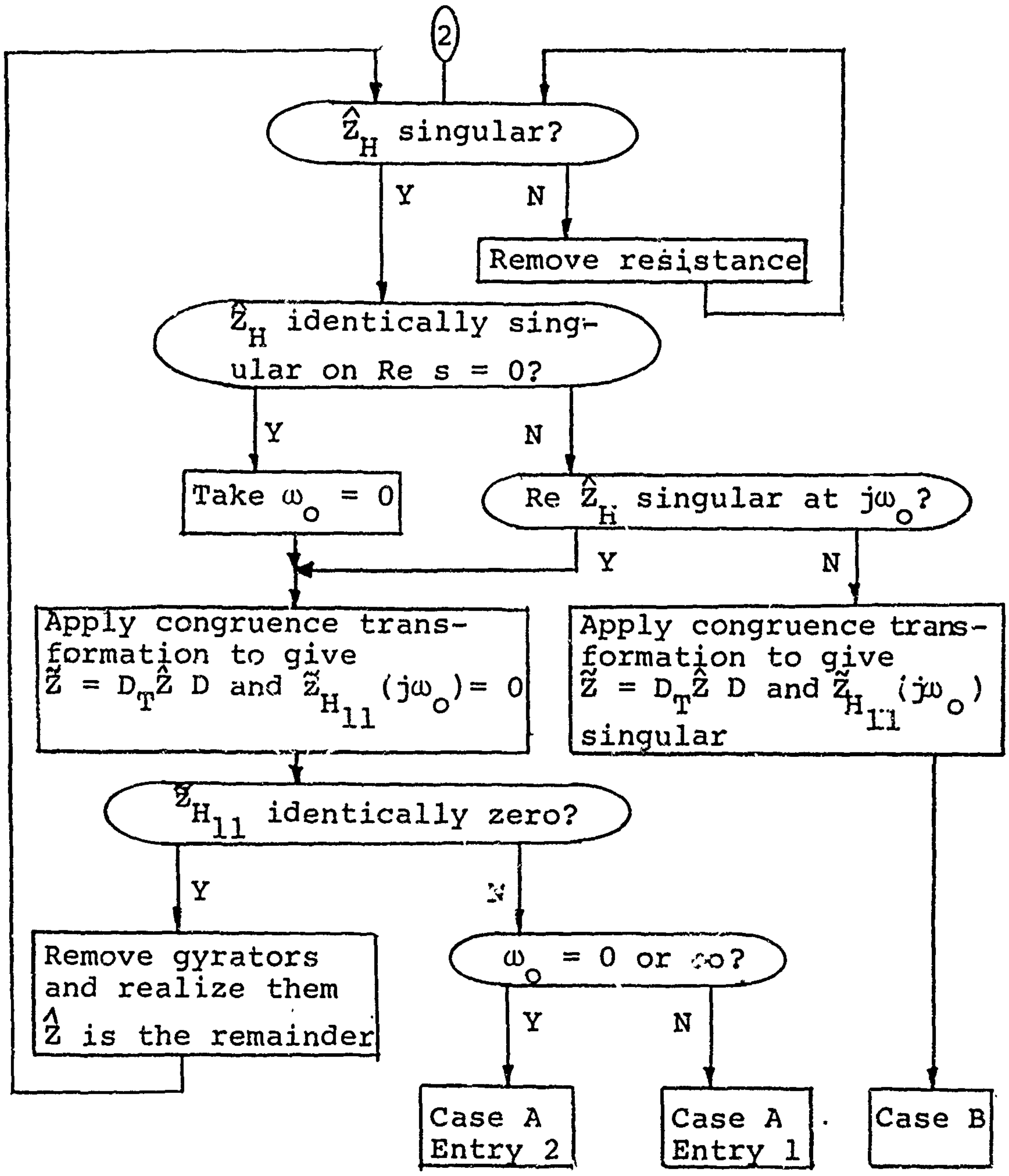
In both Cases A and B, Z" contains at most 1 gyrator. In case B, the Brune section B(s) adds at most 1 gyrator to the buffer. Hence in every case, the degree is reduced by 2 for at most 2 gyrators in the buffer.

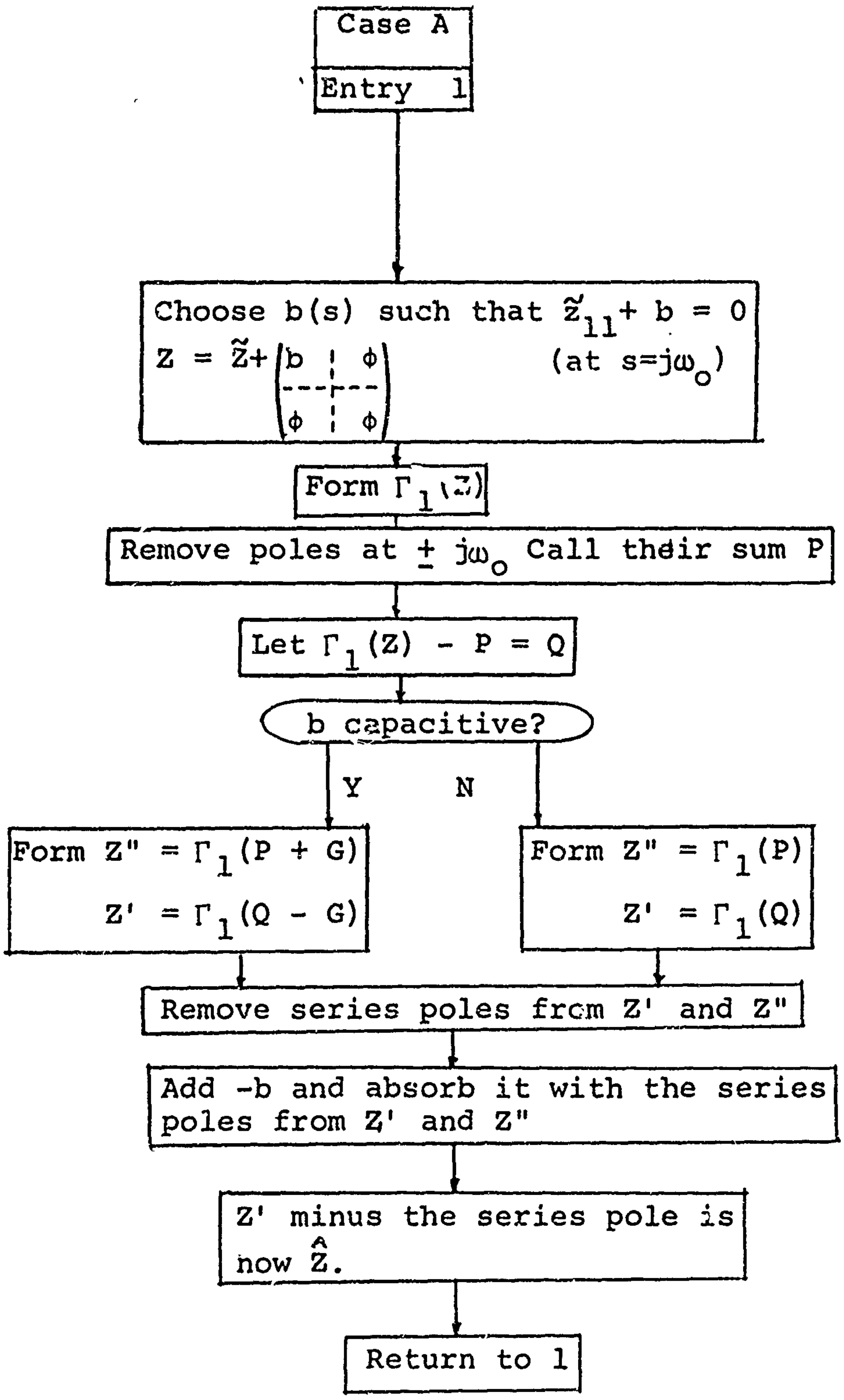
<u>This completes the proof of Theorem 22.</u>

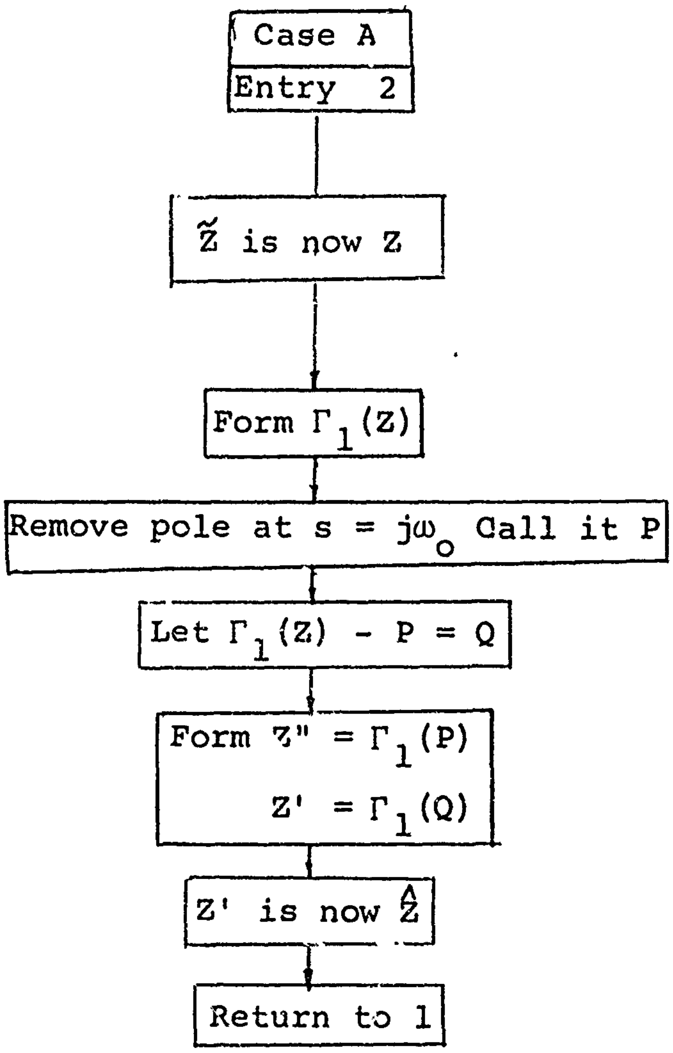
In the following pages the entire synthesis procedure is illustrated in flow chart form. The various steps are in accord with the lemmas and theorems of this chapter.

Synthesis of $\hat{Z}$




2
$\hat{Z}_H$ singular?
Y
N
Remove resistance
$\hat{Z}_H$ identically singular on Re s = 0?
Y
N
Take $\omega_o = 0$
Re $\hat{Z}_H$ singular at $j\omega_o$?
Y
N
Apply congruence transformation to give $\tilde{Z} = D_T\hat{Z}$ D and $\tilde{Z}_{H_{11}}(j\omega_o) = 0$
Apply congruence transformation to give $\tilde{Z} = D_T\hat{Z}$ D and $\tilde{Z}_{H_{11}}(j\omega_o)$ singular
$\tilde{Z}_{H_{11}}$ identically zero?
Y
N
Remove gyrators and realize them $\hat{Z}$ is the remainder
$\omega_o = 0$ or ∞?
Y
N
Case A Entry 2
Case A Entry 1
Case B


Case A
Entry 1
Choose b(s) such that $\tilde{z}_{11} + b = 0$
$Z = \tilde{Z} + \begin{pmatrix} b & \phi \\ \phi & \phi \end{pmatrix}$ (at $s = j\omega_o$)
Form $\Gamma_1(Z)$
Remove poles at $\pm j\omega_o$ Call their sum P
Let $\Gamma_1(Z) - P = Q$
b capacitive?
Y
N
Form $Z'' = \Gamma_1(P + G)$
$Z' = \Gamma_1(Q - G)$
Form $Z'' = \Gamma_1(P)$
$Z' = \Gamma_1(Q)$
Remove series poles from Z' and Z"
Add -b and absorb it with the series poles from Z' and Z"
Z' minus the series pole is now $\hat{Z}$.
Return to 1


Case A
Entry 2
$\tilde{Z}$ is now Z
Form $\Gamma_1(Z)$
Remove pole at $s = j\omega_o$ Call it P
Let $\Gamma_1(Z) - P = Q$
Form $Z'' = \Gamma_1(P)$
$Z' = \Gamma_1(Q)$
Z' is now $\hat{Z}$
Return to 1

Case B

Find $B(s)$ such that $\tilde{Z}_{11} + B$ is singular at $j\omega_o$. $Z = \tilde{Z} + B$

Form $\Gamma_{1,2}(Z)$

Remove poles at $\pm j\omega_o$. Call their sum P

$Q = \Gamma_{1,2}(Z) - P$

Add G to P so that $Z'' = \Gamma_{1,2}(P+G)$ has a gyrator portion of rank 2.

$Z' = \Gamma_{1,2}(Q-G)$

Remove series poles from Z' and Z'' Use them to absorb $-B$

$\hat{Z}$ is now Z' minus the series pole

Return to 1

Example of Case B Synthesis

In order to demonstrate the Case B synthesis procedure, we have selected a matrix which already has the property that the hermitian part of its first principal 2 x 2 submatrix is singular on the j-axis. Thus the initial removal of resistance is not necessary in this case. Clearly no generality has been lost by selecting such a pre-conditioned matrix.

We wish to synthesize $\tilde{Z}$ =

$$\begin{pmatrix} \dfrac{37s^2 + 43s + 10}{(s+1)(s+2)} & 6\,\dfrac{s+1}{s+2} & \dfrac{13s+14}{s+2} \\[2ex] 6\,\dfrac{s}{s+2} & \dfrac{s+1}{s+2} & 2\,\dfrac{s+1}{s+2} \\[2ex] \dfrac{11s-2}{s+2} & 2\,\dfrac{s+1}{s+2} & \dfrac{5s+6}{s+2} \end{pmatrix}$$

Computing $\tilde{Z}$ at s = j we obtain

$$\tilde{Z}(j) = \frac{1}{5}\begin{pmatrix} 51 + j62 & 6(3 + j) & 41 + j12 \\ 6(1 + j2) & 3 + j & 2(3 + j) \\ 7 + j24 & 2(3 + j) & 17 + j4 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 51 & 12-j3 & 24-j6 \\ 12+j3 & 3 & 6 \\ 24+j6 & 6 & 17 \end{pmatrix} + \frac{j}{5}\begin{pmatrix} 62 & 9-j6 & 18-j17 \\ 9+j6 & 1 & 2 \\ 18+j17 & 2 & 4 \end{pmatrix}$$

$$= \tilde{Z}_H(j) + \tilde{Z}_{SH}(j).$$

Hence

$$\tilde{Z}_{H_{11}}(j) = \frac{1}{5}\begin{pmatrix} 51 & 12-j3 \\ 12+j3 & 3 \end{pmatrix}$$

is singular, and so for this case $\omega_o = 1$.

<u>Brune Section</u>

In order to make $\tilde{Z}_{11}(j)$ singular it is necessary to add a Brune section which cancels $\tilde{Z}_{SH_{11}}(j)$.

In Lemma 9 an existence theorem is given which shows that a matrix B exists such that $\tilde{Z}_{11} + B$ is singular at $s = j$. In what follows the matrix B is obtained in a slightly different fashion, one which is perhaps computationally more convenient.

$B(s)$ will be so chosen that $B(j)$ cancels the matrix $Z_{SH_{11}}(j)$. To give $B(s)$ its required properties we also add an undetermined amount of the matrix $Z_{H_{11}}(j)$. The determination of the multiplier finally fixes B and is done in such a way that B has a rank 1 pole at ∞.

$$B(j) = -\frac{j}{5}\begin{pmatrix} 62 & 9-j6 \\ 9+j6 & 1 \end{pmatrix} - j\,\frac{\lambda}{5}\begin{pmatrix} 51 & 12-j3 \\ 12+j3 & 3 \end{pmatrix}$$

where λ is chosen so that $B(s)$ has a symmetric portion comprised of a rank 1 pole at ∞, plus a constant skew-symmetric matrix. The symmetric part of $B(j)$ is

$$\frac{B(j) + B(j)_T}{2} = -\frac{j}{5}\begin{pmatrix} 62+51\lambda & 9+12\lambda \\ 9+12\lambda & 1+3\lambda \end{pmatrix}$$

which must be made singular by the choice of λ.

i.e. $(153\lambda^2+237\lambda+62) - (144\lambda^2+216\lambda+81) = 0$

This gives

$$\lambda = \frac{-7 \pm 5\sqrt{5}}{6}$$

Using $\lambda = \frac{-7-5\sqrt{5}}{6}$ gives

$$B(j) = j\begin{pmatrix} \frac{17\sqrt{5}-1}{2} & 1+2\sqrt{5} \\ 1+2\;5 & \frac{1+\sqrt{5}}{2} \end{pmatrix} + \begin{pmatrix} 0 & \frac{\sqrt{5}-1}{2} \\ -\frac{(\sqrt{5}-1)}{2} & 0 \end{pmatrix}$$

i.e. we add to Z the PR matrix

$$B(s) = s\begin{pmatrix} \frac{17\sqrt{5}-1}{2} & 1+2\sqrt{5} \\ 1+2\sqrt{5} & \frac{1+\sqrt{5}}{2} \end{pmatrix} + \begin{pmatrix} 0 & \frac{\sqrt{5}-1}{2} \\ -\frac{(\sqrt{5}-1)}{2} & 0 \end{pmatrix}$$

Letting $\tilde{Z} + B = Z$ we obtain

$$z_{11}(j) = \left[\frac{1}{5} + \frac{j}{5}\left(\frac{7+5\sqrt{5}}{6}\right)\right]\begin{pmatrix} 51 & 12-j3 \\ 12+j3 & 3 \end{pmatrix}$$

which is singular as required. Z(s) follows.

$$
Z(S) = \left(\begin{array}{cc|c}
\dfrac{S^3(17\sqrt{5}-1)+S^2(71+51\sqrt{5})+S(84+34\sqrt{5})+20}{2(S+1)(S+2)} & \dfrac{S^2(2+4\sqrt{5})+S(15+9\sqrt{5})+(10+2\sqrt{5})}{2(S+2)} & \dfrac{13S+14}{S+2} \\[2ex]
\dfrac{S^2(2+4\sqrt{5})+S(17+7\sqrt{5})+(2-2\sqrt{5})}{2(S+2)} & \dfrac{S^2(1+\sqrt{5})+S(4+2\sqrt{5})+2}{2(S+2)} & \dfrac{2(S+1)}{S+2} \\[2ex]
\hline
\dfrac{11S-2}{S+2} & \dfrac{2(S+1)}{S+2} & \dfrac{5S+6}{S+2}
\end{array}\right)
$$

Z_{11} is singular at $s = j$, and so $\Gamma_{1,2}(Z)$ will have a pole at $s = j$. Let

$$U = \begin{pmatrix} I_2 \\ \hline Z_{21} \end{pmatrix} Z_{11}^{-1} \left(I \;\vdots\; -Z_{12} \right)$$

Then we can extract the pole matrix from U. The residue matrix in Z_{11}^{-1} at $s = j$ is

$$(s - j)Z_{11}^{-1}\Big|_{s=j} = \frac{3-\sqrt{5}}{4} \begin{pmatrix} 1 & -4+j \\ -4-j & 17 \end{pmatrix}$$

Thus the residue matrix of the pole in U at $s = j$ is

$$V = (s - j)U\Big|_{s=j}$$

$$= \frac{3-\sqrt{5}}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{7+j24}{5} & \frac{6+j2}{5} \end{pmatrix} \begin{pmatrix} 1 & -4+j \\ -4-j & 17 \end{pmatrix} \begin{pmatrix} 1 & 0, & -\frac{41+j12}{5} \\ 0 & 1 & -\frac{6+j2}{5} \end{pmatrix}$$

$$= \frac{3-\sqrt{5}}{4}\begin{pmatrix} 1 & -4+j & -3-j2 \\ -4-j & 17 & 10+j11 \\ -3+j2 & 10-j11 & 13 \end{pmatrix}$$

which is of rank 1.

The pole at s = -j in U has a conjugate residue V*

The entire pole matrix removable from $\Gamma_{1,2}(Z)$ is thus

$$P(s) = \frac{V}{s-j} + \frac{V^*}{s+j}$$

$$= \frac{3-\sqrt{5}}{9}\;\frac{\begin{pmatrix} 2s & -(8s+2) & -(6s-4) \\ -(8s-2) & 34s & 20s-22 \\ -(6s+4) & 20s+22 & 26s \end{pmatrix}}{s^2+1}$$

Letting $Z''(s) = \Gamma_{1,2}(P)$ we obtain $Z'' =$

$$\frac{1}{3-\sqrt{5}}\begin{pmatrix} 34s & 8s+2 & 11(3-\sqrt{5}) \\ 8s-2 & 2s & 2(3-\sqrt{5}) \\ -11(3-\sqrt{5}) & -2(3-\sqrt{5}) & 0 \end{pmatrix}$$

$$\text{i.e. } Z''(s) = \frac{3+\sqrt{5}}{2}\, s \begin{pmatrix} 17 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{3+\sqrt{5}}{2} & 11 \\ \frac{-3-\sqrt{5}}{2} & 0 & 2 \\ -11 & -2 & 0 \end{pmatrix}$$

which is PR of degree 2, with 1 gyrator.

We now form $\Gamma_{1,2}(Z)$ which is given on the following page. $Q = \Gamma_{1,2}(Z) - P$ is given on the page following $\Gamma_{1,2}(Z)$.

$$
\Gamma'_{1,3}[Z(S)] = \frac{\begin{pmatrix} S^2(1+\sqrt{5})+S(4+2\sqrt{5})+2 & -[S^2(2+4\sqrt{5})+S(15+9\sqrt{5})+(10+2\sqrt{5})] & \dfrac{-[S^3(9+5\sqrt{5})+S^2(32+14\sqrt{5})+S(32+6\sqrt{5})+8-4\sqrt{5}]}{S+2} \\ -[S^2(2+4\sqrt{5})+S(17+7\sqrt{5})+2-2\sqrt{5}] & \dfrac{[S^3(17\sqrt{5}-1)+S^2(71+5\sqrt{5})+S(84+34\sqrt{5})+20]}{S+1} & \dfrac{[S^3(28+18\sqrt{5})+S^2(107+45\sqrt{5})+S(95+4\sqrt{5})-(12+28\sqrt{5})]}{S+1} \\ \dfrac{[S^3(7+3\sqrt{5})+S^2(4-2\sqrt{5})-S(24+14\sqrt{5})-(8-4\sqrt{5})]}{S+2} & \dfrac{[-S^3(24+10\sqrt{5})+S^2(-19+11\sqrt{5})+S(88+64\sqrt{5})+60+4\sqrt{5}]}{S+2} & 0 \end{pmatrix}_{(S+1)(S+2)}}{(S^2+1)[S^2(12+6\sqrt{5})+S(34+16\sqrt{5})+20+8\sqrt{5}]}
$$

$$
+ \frac{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & [S^4(17+11\sqrt{5})+S^3(101+57\sqrt{5})+S^2(216+94\sqrt{5})+S(192+44\sqrt{5})+(56-8\sqrt{5})] \end{pmatrix}}{(S^2+1)(S^2(12+6\sqrt{5})+S(34+16\sqrt{5})+20+8\sqrt{5})}
$$

$$\Gamma_{1,2}[Z(S)] - P(S) = \frac{\begin{pmatrix} S^2(1+\sqrt{5})+S(4+2\sqrt{5})+4 & -[S^2(2+4\sqrt{5})+S(9+5\sqrt{5})+(10+2\sqrt{5})] & -[S^2(9+5\sqrt{5})+S(32+10\sqrt{5})+28] \\ -[S^2(2+4\sqrt{5})+S(11+7\sqrt{5})+14-2\sqrt{5}] & S^2(-1+17\sqrt{5})+S(18+34\sqrt{5})+40 & S^2(28+18\sqrt{5})+S(105+33\sqrt{5})+98-6\sqrt{5} \\ S^2(7+3\sqrt{5})+S(20+10\sqrt{5})+12+8\sqrt{5} & -[S^2(24+10\sqrt{5})+S(73+29\sqrt{5})+50+18\sqrt{5}] & S^2(17+11\sqrt{5})+S(52+18\sqrt{5})+58-8\sqrt{5} \end{pmatrix}}{S^2(12+6\sqrt{5})+S(34+18\sqrt{5})+20+8\sqrt{5}}$$

Then $Z'(s) = \Gamma_{1,2}(Q) =$

$$\left(\begin{array}{c:c:c} s(62+29\sqrt{5}) + 35+15\sqrt{5} & s\frac{(37+17\sqrt{5})}{2} + \frac{25+11\sqrt{5}}{2} & 2-3\sqrt{5} \\ \hdashline s\frac{(37+17\sqrt{5})}{2} + \frac{17+7\sqrt{5}}{2} & s\frac{(11+5\sqrt{5})}{2} + \frac{7+3\sqrt{5}}{2} & -1-\sqrt{5} \\ \hdashline 4+5\sqrt{5} & 1+\sqrt{5} & 5 \end{array}\right)$$

$$= s\begin{pmatrix} 62+29\sqrt{5} & \frac{37+17\sqrt{5}}{2} & 0 \\ \frac{37+17\sqrt{5}}{2} & \frac{11+5\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 35+15\sqrt{5} & \frac{21+9\sqrt{5}}{4} & 3+\sqrt{5} \\ \frac{21+9\sqrt{5}}{4} & \frac{7+3\sqrt{5}}{2} & 0 \\ 3+\sqrt{5} & 0 & 5 \end{pmatrix}$$

$$+\begin{pmatrix} 0 & \frac{2+\sqrt{5}}{2} & -1-4\sqrt{5} \\ \frac{-2-\sqrt{5}}{2} & 0 & -1-\sqrt{5} \\ 1+4\sqrt{5} & 1+\sqrt{5} & 0 \end{pmatrix}$$

The first of these three terms is a rank 1 pole at ∞ and can be realized by a single inductor and ideal transformers. The second term is purely resistive and can be realized by 3 resistors and ideal transformers, and the third term can be realized by a single gyrator and ideal transformers.

The rank 1 pole at ∞ in Z' is called M. It is caused by the addition of B to $\tilde{Z}$.

From $Z''(s)$ we obtain a series pole at ∞ given by

$$N = s\,\frac{3+\sqrt{5}}{2}\begin{pmatrix} 17 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The connection sequence of Z', Z'', M, N and -B is shown in Figures 29, 30, 31 and 32.

The matrices -B, N and M of Figure 32 are combined by the method given in the proof of Theorem 21 into a matrix

$$T = \left(\begin{array}{c|c} -B + N & N \\ \hline N & N + M \end{array}\right)$$

The reactive part of T is

$$s\begin{pmatrix} 26 & 5 & \frac{51+17\sqrt{5}}{2} & \frac{12+4\sqrt{5}}{2} \\ 5 & 1 & \frac{12+4\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} \\ \frac{51+17\sqrt{5}}{2} & \frac{12+4\sqrt{5}}{2} & \frac{175+75\sqrt{5}}{2} & \frac{49+21\sqrt{5}}{2} \\ \frac{12+4\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} & \frac{49+21\sqrt{5}}{2} & \frac{14+6\sqrt{5}}{2} \end{pmatrix}$$

which is rank 2 and may be realized by congruence transformer and 2 inductors. The final realization for Z is shown in Figure 33.

APPENDIX I

The PR Property

A. THE IMPEDANCE MATRIX.

Definition 1.1 A network is said to be *passive* if there are no energy sources within that network.

Suppose a network is excited by an external energy source. Any increase of the energy within the network must be obtained from that external source. We thus have

Axiom 1.1 A passive network cannot posess an increasing amount of energy when all external forcing functions are zero.

Suppose that a passive network is excited by an external forcing function, which is periodic (period T). Once a steady state has been achieved the transfer of average energy can only be from the exciting source into the network. We state this as

Axiom 1.2 The average energy supplied to a passive network by a periodic source is, in the steady state, non-negative.

Let e(t) be the instantaneous energy entering a passive network and p(t) the instantaneous power. Then

$$p(t) = \frac{d\, e(t)}{dt}$$

Let the source be periodic and let a steady state exist. Then

$$P_{ave} = \frac{1}{T} \int_{nT}^{(n+1)T} p(t)\, dt$$

$$= \frac{1}{T}[e((n+1)T) - e(nT)].$$

By Axiom 1.2, $e((n+1)T) \geq e(nT)$. Hence we have the following

<u>Theorem 1.1</u> <u>The average power entering a passive network from a periodic source is, in the steady state, non-negative</u>.

Suppose that there are n ports at which forcing functions may be applied, and let the responses be measured at those n ports. Let the network consist of finite numbers of resistors,

capacitors, gyrators, ideal transformers, self and mutual inductors, all of which are linear. Let the forcing functions at the n ports be currents, and the responses be voltages.

Then $v_k(t)$, the voltage at the k-th port is related to $i_r(t)$, the current entering the r-th port by a linear differential equation of the form

$$(1.1) \qquad \sum_{m=0}^{p} \alpha_{km} D^m v_k(t) = \sum_{v=0}^{q} \beta_{rv} D^v i_r(t)$$

where D means $\frac{d}{dt}$.

The coefficients α_{km} and β_{rv} are obtained by the application of Kirchhoff's laws to the network. Since a network comprised of a finite number of elements can have only a finite number of loops and nodes, it follows that p and q are finite. Moreover, if all of the elements in the network are physically realizable, then their values are given by real numbers. The application of Kirchhoff's equations will encounter only real numbers and so all of the α_{km} and β_{rv} will be real.

Suppose we restrict the forcing functions $i_r(t)$ to be in the class of functions which are Laplace transformable. Assuming no stored energy within the network at t = 0, we obtain, by transforming both sides of Equation 1.1,

$$V_k(s) = \frac{\sum_{v=0}^{q} \beta_{rv} s^v}{\sum_{m=0}^{p} \alpha_{km} s^m} I_r(s)$$

where s is the Laplace complex variable.

Define

$$z_{kr} = \frac{\sum_{v=0}^{q} \beta_{rv} s^v}{\sum_{m=0}^{p} \alpha_{km} s^m}$$

z_{kr} is a rational function in s with real coefficients.

By the assumption of linearity if currents are simultaneously applied to all n ports,

$$V_k(s) = \sum_{r=1}^{n} z_{kr}(s) I_r(s) \qquad k=1,2\ldots n$$

This gives the general equation for the network as

$$V(s)\rangle = Z(s)I(s)\rangle ,$$

where $V(s)\rangle$ is the vector of Laplace transforms of the port voltages and $I(s)\rangle$ the vector of port currents. $Z(s)$ is an n x n matrix called the driving point impedance matrix. We have thus proved the following

Theorem 1.2 Let $Z(s)$ be the driving point impedance matrix of a passive linear finite lumped (PLFL) electrical network. Then the elements of $Z(s)$ are rational functions of s with real coefficients.

B. CONSTRAINTS ON THE POLES OF z_{ij}.

Let all currents be zero, except for $i_r(t)$. Suppose that $i_r(t)$ is a rectangular pulse of finite duration starting at $t = 0$, and such that after $t = t_1$, $i_r(t) = 0$. If $z_{kr}(s)$ posesses poles in Re $s > 0$, then it can be shown that after $t = t_1$, when all forcing functions are zero, $v_k(t)$ increases without bound. Since this implies an increase in the internal energy of the system, a violation of Axiom 1.1 is

encountered. Hence $z_{kr}(s)$ cannot have poles in Re $s > 0$, and this is true for $k,r = 1,2...n$.

Assume $z_{kr}(s)$ has a pole on the j-axis at $s = j\omega_o$, and apply the same current pulse. If the pole has a multiplicity equal to m then $v_k(t)$ will have in it a term $f(t)\cos(\omega_o t+\beta)$ where $f(t)$ is a polynomial of degree m-1. Clearly if $m > 1$, this term will have an amplitude which increases without bound, again violating Axiom 1.1. We thus have

Theorem 1.3 *Let Z be the impedance matrix of a PLFL network. Then the z_{ij} have no poles in Re $s > 0$ and any poles on Re $s = 0$ are of single multiplicity.*

C. ENERGY CONSIDERATIONS

Suppose a network whose impedance matrix is $Z(s)$, is excited by a current vector $i(t)\}$ where $i_r(t)$ is entering at the r-th port. Let $i(t)\}$ be of the form

$$i(t)\rangle = \begin{pmatrix} m_1\cos(\omega_o t+\alpha_1) \\ m_2\cos(\omega_o t+\alpha_2) \\ \vdots \\ m_n\cos(\omega_o t+\alpha_n) \end{pmatrix}$$

where m_k, α_k and ω_o are all real, and where $\omega_o \neq 0$ or ∞. Note that all components of $i(t)\rangle$ are sinusoidal with the same frequency. Phases α_k and amplitudes m_k are completely arbitrary.

$i(t)\rangle$ may be written

$$i(t)\rangle = \mathrm{Re}\left\{ e^{j\omega_o t} \begin{pmatrix} m_1 e^{j\alpha_1} \\ m_2 e^{j\alpha_2} \\ \vdots \\ m_n e^{j\alpha_n} \end{pmatrix} \right\}$$

Let $I\rangle = \begin{pmatrix} m_1 e^{j\alpha_1} \\ \vdots \\ m_n e^{j\alpha_n} \end{pmatrix} = a\rangle + jb\rangle$

where $a\downarrow$ and $b\downarrow$ are real n-vectors. Then $i(t)\downarrow = \mathrm{Re}(e^{j\omega_o t} I\downarrow)$.

Let $\eta(t)\downarrow = e^{j\omega_o t} I\downarrow$. Then letting $H(s)\downarrow$ be the Laplace transform of $\eta(t)\downarrow$, we have

$$H(s)\downarrow = \frac{I\downarrow}{s - j\omega_o}$$

If $H(s)\downarrow$ is applied to the ports of $Z(s)$ the corresponding voltages will be

$$N(s)\downarrow = Z(s)H(s)\downarrow$$

$$= \frac{Z(s)}{s - j\omega_o} I\downarrow$$

Assume that $j\omega_o$ is not a pole of $Z(s)$. By Theorem 1.2, j-axis poles in $z_{ij}(s)$ are simple, and poles in the right half plane do not exist. Hence letting $n(t)\downarrow$ be the inverse Laplace transform of $N(s)\downarrow$ we obtain

$$n(t)\rangle = Z(j\omega_o)e^{j\omega_o t}\, I\rangle + \sum_k \frac{W_k e^{j\omega_o t}}{j(\omega_k - \omega_o)}\, I\rangle$$

$$+ \sum_r F_r(t)e^{(\sigma_r + j\alpha_r)t}\, I\rangle$$

where W_k is a residue matrix at a possible j-axis pole of $Z(s)$ and where $F_r(t)$ is a matrix of polynomials in t of order one less than the multiplicity of the r-th pole of $z_{ij}(s)$ in the left half plane at $s = \sigma_r + j\omega_r$, $(\sigma_r < 0)$.

In the <u>steady state</u>, (t very large)

$$n(t)\rangle = Z(j\omega_o)e^{j\omega_o t}\, I\rangle + \sum_k \frac{W_k e^{j\omega_o t}}{j(\omega_k - \omega_o)}\, I\rangle$$

$n(t)\rangle$ has a real and an imaginary part. The response to the real part of $\eta(t)\rangle$ must be the real part of $n(t)\rangle$.

Hence if $i(t)\rangle = \mathrm{Re}(e^{j\omega_o t}\, I\rangle)$ we obtain

$$v(t)\rangle = \mathrm{Re}\{Z(j\omega_o)e^{j\omega_o t}\, I\rangle + \sum_k \frac{W_k e^{j\omega_o t}}{j(\omega_k - \omega_o)}\, I\rangle\}$$

The instantaneous steady state power into the network is

$$p_{ss}(t) = \sum_{m=1}^{n} i_m(t) v_m(t)$$

$$= \mathrm{Re}(e^{j\omega_o t} \vec{I}) \; \mathrm{Re}\{Z(j\omega_o) e^{j\omega_o t} I\downarrow\}$$

$$+ \mathrm{Re}(e^{j\omega_o t} \vec{I}) \; \mathrm{Re}\{\sum_k \frac{W_k e^{j\omega_o t}}{j(\omega_k - \omega_o)} I\downarrow\}$$

Recall that $I\downarrow = a\downarrow + jb\downarrow$. Let

$$Z(j\omega_o) I\downarrow = E\downarrow = c\downarrow + jd\downarrow$$

and

$$\frac{W_k I\downarrow}{j(\omega_k - \omega_o)} = e_k\downarrow + jf_k\downarrow .$$

Then

$$p_{ss}(t) = \mathrm{Re}\{(\vec{a} + j\vec{b}) e^{j\omega_o t}\} \; \mathrm{Re}\{(c\downarrow + jd\downarrow) e^{j\omega_o t}\}$$

$$+ \mathrm{Re}\{(\vec{a} + j\vec{b}) e^{j\omega_o t}\} \; \mathrm{Re}\{\sum_k (e_k\downarrow + jf_k\downarrow) e^{j\omega_o t}\}$$

$$= \vec{a}c\downarrow \cos^2\omega_o t + \vec{b}d\downarrow \sin^2\omega_o t$$

$$- (\vec{a}d\downarrow + \vec{b}c\downarrow) \sin \omega_o t \cos \omega_o t$$

$$+ (\vec{a} \cos \omega_o t - \vec{b} \sin \omega_o t) \sum_k (e_k\downarrow \cos \omega_k t - f_k\downarrow \sin \omega_k t)$$

The average power will be

$$P_{ave} = \frac{\omega_o}{2\pi} \int_o^{2\pi/\omega_o} p_{ss}(t)dt$$

$$= \frac{1}{2}(\vec{a}c\!\downarrow + \vec{b}d\!\downarrow)$$

Now,

$$\frac{1}{2}\,\mathrm{Re}(\overrightarrow{I^*}E\!\downarrow) = \frac{1}{2}\,\mathrm{Re}\Big[(\vec{a} - j\vec{b})(c\!\downarrow + jd\!\downarrow)\Big]$$

$$= \frac{1}{2}(\vec{a}c\!\downarrow + \vec{b}d\!\downarrow).$$

Thus

$$P_{ave} = \frac{1}{2}\,\mathrm{Re}(\overrightarrow{I^*}E\!\downarrow).$$

But $E\!\downarrow = Z(j\omega_o)I\!\downarrow$. Hence

$$P_{ave} = \frac{1}{2}\,\mathrm{Re}(\overrightarrow{I^*}Z(j\omega_o)I\!\downarrow).$$

But by Theorem 1.1, $P_{ave} \geq 0$. Hence we must have, for ω_o not a pole of $Z(s)$ and for $0 < \omega_o < \infty$,

$$\mathrm{Re}(\overrightarrow{I^*}Z(j\omega_o)I\!\downarrow) \geq 0.$$

If $\omega_o = 0$, $\eta(t)\downarrow = I\downarrow$ which is real and $n(t)\downarrow = Z(0)I\downarrow$ which is also real. Hence

$$p_{ss}(t) = \mathrm{Re}(\vec{I})\ \mathrm{Re}(Z(0)I\downarrow)$$

$$+ \text{ terms due to possible j-axis poles in } z_{ij}(s)$$

$$= \mathrm{Re}\ \vec{a}\ \mathrm{Re}\ c\downarrow$$

$$= \vec{a}c\downarrow$$

Hence $P_{ave} = \vec{a}c\downarrow$, and

$$\mathrm{Re}(\vec{I}^*E\downarrow) = \mathrm{Re}(\vec{a}c\downarrow) = \vec{a}c\downarrow$$

Thus as before, for 0 not a pole of $z_{ij}(s)$,

$$\mathrm{Re}(\vec{I}^*Z(0)I\downarrow) \geqq 0.$$

If $\omega_o = \infty$ we apply the transformation $s' = \frac{1}{s}$. Then $s = \infty$ maps into $s' = 0$. An inductor sL is now replaced by a capacitor $\frac{1}{s'(\frac{1}{L})}$ and a capacitor $\frac{1}{sC}$ is replaced by an inductor $s'(\frac{1}{C})$. Clearly this is a passive network and as before $P_{ave} \geqq 0$, which implies

$$\mathrm{Re}(\overrightarrow{I^*}Z(s)I\downarrow) \geq 0 \text{ at } s' = 0$$

We have thus proved

Theorem 1.4 *If* $Z(s)$ *is the driving point impedance matrix of a* PLFL *network and* $j\omega_o$ *is not a pole of* Z, *then for any complex vector* $\vec{x}$,

$$\mathrm{Re}(\overrightarrow{x^*}Z(j\omega_o)x\downarrow) \geq 0.$$

Corollary 1.4.1 *The hermitian part of* $Z(s)$ *is non-negative definite everywhere on the* j-axis *where* Z *has no poles*.

Proof.

$$\mathrm{Re}(\overrightarrow{x^*}Z(s)x\downarrow) = \overrightarrow{x^*}Z_H(s)x\downarrow \quad \text{for all } \vec{x}$$

Corollary 1.4.2 *If* $Z(s)$ *is the impedance matrix of a lossless network (i.e. without resistors) then* $Z_H(s)$ *is null everywhere on the* j-axis *where* Z *has no poles*.

Proof. Since the average power delivered to the network is zero,

$$\mathrm{Re}(\overrightarrow{x^*}Z(j\omega_o)x\downarrow) = 0$$

for all $\vec{x}$.

D. THE ASSOCIATE FUNCTIONS

Definition 1.2 Let $f(s)$ be a rational function of the complex variable s, with real coefficients. Then $f(s)$ is said to be a real rational function of s.

Definition 1.3 Let $F(s)$ be a matrix whose elements are rational functions of s. Then F is said to be a rational matrix function of s. F may be a rational matrix function which is any or all of square, real, symmetric etc.

Definition 1.4 Let F be a square real rational n x n matrix function and let $\vec{x}$ be a constant complex n-vector. Then $f(s) = \vec{x}^*F\vec{x}$ is called an associate function of F. Note that for any F there are arbitrarily many associate functions.

If F is symmetric then f is a real rational function for any $\vec{x}$. If $\vec{x}$ is real then f is a real rational function even if F is not symmetric.

In general however, f is a rational function with complex coefficients.

Theorem 1.5 Let $Z(s)$ be the matrix of a PLFL electrical network. Let $z(s)$ be an associate

function of z. *Then*

i) Re $z(s) \geq 0$ *for* Re $s = 0$

ii) $z(s)$ *has no poles in* Re $s > 0$

iii) *If* $z(s)$ *has poles on* Re $s = 0$ *then they are of single multiplicity, and have positive real residues.*

Proof.

Proposition i) follows immediately from Theorem 1.4.

By Theorem 1.3, none of the z_{ij} have poles in Re $s > 0$. Now $z = \sum_{i=1}^{n} \sum_{j=1}^{n} z_{ij} x_i^* x_j$, and so $z(s)$ cannot have poles in Re $s > 0$. Hence ii) is proved.

By Theorem 1.3, if any of the z_{ij} have poles on Re $s = 0$, then they are of simple multiplicity. Hence $z(s)$ can have poles on Re $s = 0$ only if they are simple.

Suppose $z(s)$ has a pole at $j\omega_o$. Let $s - j\omega_o = \rho e^{j\theta}$. Then for very small ρ,

$$z(s) \doteq \frac{Ke^{j\alpha}}{\rho e^{j\theta}}$$

where $Ke^{j\alpha}$ is the residue at the pole. Since Re $z(s) \geq 0$ on $s = j\omega$ (by i) we have

$$\cos(\alpha + \theta) \geq 0 \quad \text{for } \theta = \pi/2, \ -\pi/2.$$

But this can only be true if $\alpha = 0$, i.e. $z(s)$ has a positive real residue at a j-axis pole. This completes iii) and proves the theorem.

Corollary 1.5.1 If Z, the matrix of a PLFL network, has a pole on the j-axis then the residue is a non-negative definite hermitian matrix.

Proof.

Let $z(s) = \overrightarrow{x^*}Z(s)x\rangle$. If $Z(s)$ has a pole at $s = j\omega_o$, let $W = (s - j\omega_o)Z(s)\Big|_{s=j\omega_o}$ be the residue matrix, and let

$$w = (s - j\omega_o)z(s)\Big|_{s=j\omega_o} \quad \text{be the}$$

residue in $z(s)$. Then

$$w = \overrightarrow{x^*}Wx\rangle.$$

But if w if non-zero, then by iii) of Theorem 1.5, it must be positive and real. Hence

$$\vec{x}^*Wx\rangle \geq 0.$$

This means that W_H is non-negative definite and W_{SH} is null. Hence W is non-negative definite hermitian, <u>which proves the corrolary.</u>

E. FURTHER IMPLICATIONS

<u>Lemma 1.1</u> <u>Excluding possible</u> j-axis <u>poles, let</u> f(s)

i) <u>be requalar on</u> Re $s = 0$

ii) <u>have no poles in</u> Re $s > 0$

iii) <u>be such that</u> Re $f(s) \geq 0$ <u>on</u> Re $s = 0$.

<u>Then</u> Re $f(s) \geq 0$ <u>for</u> Re $s \geq 0$.

<u>Proof.</u>

Let C be the closed curve consisting of the j-axis except at poles of f(s) where C is a small semi-circle of radius ρ in the right half plane. Let R be the region enclosed by C, in the right half plane.

By the Principle of the Minimum, (well known in functions of a complex variable)

$$0 \leq \text{Re } f(s)\Big|_{s \in C} \leq \text{Re } f(s)\Big|_{s \in R}$$

Since ρ can be taken arbitrarily small it follows that, except at possible j-axis poles,

$$\operatorname{Re} f(s) \geqq 0 \quad \text{for } \operatorname{Re} s \geqq 0.$$

This proves the lemma.

Corollary 1.1.1 _If_ $z(s)$ _is an associate function of_ $Z(s)$, _the matrix of a_ PLFL _network, then_

$$\operatorname{Re} z(s) \geqq 0 \quad \underline{\text{for}} \operatorname{Re} s \geqq 0.$$

Proof.

By Theorem 1.5, all conditions of Lemma 1.1 are satisfied for $z(s)$.

We can now prove the following

Theorem 1.6 _Let_ $Z(s)$ _be the matrix of a_ PLFL _network._ _Then_ $Z_H(s)$ _is either hermitian positive definite in_ $\operatorname{Re} s > 0$ _or it is identically null for all_ s.

Proof.

Let $z(s)$ be an associate function of $Z(s)$. Suppose $\operatorname{Re} z(s) = 0$ at an isolated point $s = s_o$ in the right half plane, and suppose that $z(s)$ is not identically zero. i.e.

$$z(s_o) = j\beta \quad (\beta \text{ real})$$

Then

$$\frac{1}{z(s) - j\beta} = \frac{Ke^{j\alpha}}{(s - s_o)^n} + g(s)$$

where $g(s)$ has poles at s_o of order n-1 or less. Let $s - s_o = \rho e^{j\theta}$. Then for ρ sufficiently small,

$$\operatorname{Re}\left(\frac{1}{z(s) - j\beta}\right) \doteqdot \frac{K}{\rho} \cos(\alpha - n\theta)$$

Now

$$\operatorname{Re}\left(\frac{1}{z(s) - j\beta}\right) = \frac{\operatorname{Re} z(s)}{\{\operatorname{Re} z(s)\}^2 + \{\operatorname{Im} z(s) - \beta\}^2}$$

But by the corollary to Lemma 1.1, $\operatorname{Re} z(s) \geq 0$ in $\operatorname{Re} s \geq 0$. In particular, this is true on $s - s_o = \rho e^{j\theta}$. Hence

$$\cos(\alpha - n\theta) \geq 0 \quad \text{for } 0 \leq \theta \leq 2\pi.$$

Clearly then $n = 0$. But in that case $\operatorname{Re} z(s)$ does not have a zero at an isolated point in $\operatorname{Re} s > 0$, which contradicts the assumption that it does.

Since $\operatorname{Re} z(s) \geq 0$ in $\operatorname{Re} s \geq 0$, either

Re $z(s) > 0$ in Re $s > 0$ or Re $z(s) = 0$ for all s. Since this is true for any associate function of $Z(s)$ it follows that either Z_H is hermitian positive definite in Re $s > 0$ or it is null for all s. This completes the proof of the theorem.

F. THE PR PROPERTY

Definition 1.5 A square matrix function $F(s)$ is said to be positive real (PR) if I and II are satisfied.

I The matrix elements f_{ij} are rational in s with real coefficients

II For any complex vector $\vec{x}$,

$$\text{Re}(\vec{x}^* F \vec{x}) \geq 0 \quad \text{for Re } s \geq 0.$$

Theorem 1.7 Let $F(s)$ be a square real rational matrix function. Let $f(s)$ be an associate function. Then $F(s)$ is PR if and only if

IIa. Re $f(s) \geq 0$ for Re $s = 0$

IIb. $f(s)$ has no poles in Re $s > 0$

IIc. For Re $s = 0$, poles of $f(s)$ are simple and have non-negative residues.

Proof.

Suppose $F(s)$ is PR. Then for any vector $\vec{x}$,

$$\text{Re } f(s) = \text{Re}(\vec{x}^*F\vec{x}) \geq 0, \qquad \text{on Re } s = 0$$

Suppose f(s) has a pole in Re $s > 0$. i.e.

$$f(s) = \frac{Ke^{j\alpha}}{(s - s_o)^n} + g(s),$$

where g(s) has poles of order n-1 or less at s_o. Let $s - s_o = \rho e^{j\theta}$. Then for very small ρ,

$$f(s) \doteq \frac{K}{\rho} e^{j(\alpha - n\theta)}$$

and $\text{Re } f(s) = \frac{K}{\rho} \cos(\alpha - n\theta), \qquad (0 \leq \theta \leq 2\pi).$

But this is non-negative which can only be true if $n = 0$. Hence f(s) cannot have poles in Re $s > 0$.

Suppose f(s) has a pole on the j-axis. Then, following the proof of proposition iii) of Theorem 1.5, such poles of f(s) are simple and have non-negative residues.

This completes the proof of the theorem.

Corollary 1.7.1 Let F(s) be a PR matrix. If F(s) has a j-axis pole then the residue matrix is hermitian non-negative definite.

<u>Theorem 1.8</u> <u>Let</u> Z(s) <u>be the impedance matrix of a</u> PLFL <u>network.</u> <u>Then</u> Z(s) <u>is</u> PR.

<u>Proof.</u>

By Theorem 1.2 the z_{ij} are rational real functions of s.

By Theorem 1.4, for any vector $\vec{x}$,

$$\mathrm{Re}(\vec{x}^* Z x|) \geq 0 \qquad \text{on } \mathrm{Re}\, s = 0.$$

By Theorem 1.6, either

$$\mathrm{Re}(\vec{x}^* Z x|) > 0 \qquad \text{in } \mathrm{Re}\, s > 0$$

or it is identically zero. Hence

$$\mathrm{Re}(\vec{x}^* Z x|) \geq 0 \qquad \text{in } \mathrm{Re}\, s \geq 0.$$

<u>This proves the theorem and completes the appendix.</u>

APPENDIX II

Diagonalization of Skew-symmetric Matrices.

Let A be a real skew-symmetric matrix of order 2n. Then there exists an orthogonal matrix V such that

$$V_T A V = \begin{pmatrix} \begin{matrix} 0 & \mu_1 \\ -\mu_1 & 0 \end{matrix} & & & 0 \\ & \begin{matrix} 0 & \mu_2 \\ -\mu_2 & 0 \end{matrix} & & \\ & & \ddots & \\ 0 & & & \begin{matrix} 0 & \mu_n \\ -\mu_n & 0 \end{matrix} \end{pmatrix}$$

where the μ_k are real and possibly zero. This is proved as follows.

<u>Lemma 2.1</u> <u>The eigenvalues of a real skew symmetric matrix are pure imaginary, or zero.</u>

<u>Proof.</u>

Let A be real skew symmetric, let λ be an

eigenvalue with associated eigenvector $\vec{x}$.

Then

$$Ax\downarrow = \lambda x\downarrow \; . \; . \; . \; . \; . \; . \; 1.$$

and

$$Ax^*\downarrow = \lambda^* x^*\downarrow \; . \; . \; . \; 2.$$

Hence $\vec{x}^*_T A x\downarrow = \lambda \vec{x}^*_T x\downarrow$ (by 1) and $\vec{x}^*_T A x\downarrow = -\lambda^* \vec{x}^*_T x\downarrow$ by 2. Since $\vec{x}$ is not the null vector

$$\lambda = -\lambda^*. \qquad \text{QED}$$

Lemma 2.2 Let A be a real skew symmetric matrix, let $i\mu$ be an eigenvalue (μ real, nonzero) and let $x\downarrow + iy\downarrow$ be the associated eigenvector, where the vectors $x\downarrow$ and $y\downarrow$ are real. Then $x\downarrow$ and $y\downarrow$ are orthogonal.

Proof.

$$A(x\downarrow + iy\downarrow) = i\mu(x\downarrow + iy\downarrow)$$

Thus

$$Ax\downarrow = -\mu y\downarrow \; . \; . \; . \; . \; . \; . \; . \; 1.$$

and

$$Ay\downarrow = \mu x\downarrow \, . \; . \; . \; . \; . \; . \; . \; . \quad 2.$$

Hence by 2, $\vec{y}(Ay\downarrow) = -\mu\vec{x}y\downarrow$, giving $\mu\vec{y}x\downarrow = -\mu\vec{x}y\downarrow$.

But μ, $x\downarrow$ and $y\downarrow$ are real, nonzero.

Thus $\vec{x}y\downarrow = 0$. QED

<u>Lemma 2.3</u> <u>Let</u> A <u>be a real skew-symmetric</u> n x n <u>matrix,</u> $i\mu$ <u>a nonzero eigenvalue with associated</u> <u>eigenvector</u> $x\downarrow + iy\downarrow$. <u>Let</u> M <u>be a real orthogonal matrix whose first two columns are</u> $x\downarrow$ <u>and</u> $y\downarrow$ <u>Then</u>

$$M_T A\, M = \left(\begin{array}{cc|c} 0 & \mu & \multirow{2}{*}{ϕ} \\ -\mu & 0 & \\ \hline \multicolumn{2}{c|}{\phi} & A_{n-2} \end{array}\right)$$

<u>where</u> A_{n-2} <u>is a real skew symmetric matrix of</u> <u>order</u> n-2.

<u>Proof.</u>

Let $M = (x\downarrow \;\; y\downarrow \;\; m_3\downarrow \, . \; . \; . \; m_n\downarrow)$ be an orthogonal matrix. Then

$$AM = (\,Ax\downarrow \quad Ay\downarrow \quad Am_3\downarrow \quad . \; . \; . \; Am_n\downarrow\,)$$

$$= (-\mu y\downarrow \quad \mu x\downarrow \quad Am_3\downarrow \quad . \; . \; . \; Am_n\downarrow\,)$$

Now

$$\begin{aligned}\vec{x}Am_3\downarrow &= -\vec{m}_3Ax\downarrow \\ &= \mu\vec{m}_3y\downarrow \\ &= 0.\end{aligned}$$

Hence

$$M_TAM = \begin{pmatrix}\vec{x}\\ \vec{y}\\ \vec{m}_3\\ \vdots\\ \vec{m}_n\end{pmatrix}(-\mu y\downarrow \quad \mu x\downarrow \quad Am_3\downarrow \ . \ . \ . \ Am_n\downarrow)$$

$$= \left(\begin{array}{cc|c} 0 & \mu & 0. \ . \ . \ . \ .0 \\ -\mu & 0 & 0. \ . \ . \ . \ .0 \\ \hline 0 & 0 & \\ & & A_{n-2} \\ 0 & 0 & \end{array}\right)$$

Since A is skew-symmetric, so is $V_TA\ V$. Hence so is A_{n-2}.

QED

<u>Theorem 2.1</u> <u>Let</u> A <u>be a real skew-symmetric matrix of order</u> 2n. <u>Then there exists an orthogonal matrix</u> V <u>such that</u>

$$V_T A V = \begin{pmatrix} 0 & \mu_1 & & & & & 0 \\ -\mu_1 & 0 & & & & & \\ & & 0 & \mu_2 & & & \\ & & -\mu_2 & 0 & & & \\ & 0 & & & \ddots & & \\ & & & & & 0 & \mu_n \\ & & & & & -\mu_n & 0 \end{pmatrix}$$

<u>where the</u> μ_i <u>are real and some possibly zero</u>.

<u>Proof.</u>

If A has only zero eigenvalues then it is the null matrix and we take V as the identity matrix, and the proof is complete. If A has a nonzero eigenvalue, let it be called $i\mu_1$. Then by Lemma 2.3, there exists an orthogonal matrix M_1 such that

$$M_1^T A\, M_1 = \left(\begin{array}{cc|c} 0 & \mu_1 & \\ -\mu_1 & 0 & \phi \\ \hline \multicolumn{2}{c|}{\phi} & A_{2(n-1)} \end{array}\right)$$

Suppose, for a proof by induction, that there exists an orthogonal matrix M_r such that

$$M_r^T A\, M_r = \left(\begin{array}{cccccc} 0 & \mu_1 & & & & \\ -\mu_1 & 0 & & & & 0 \\ & & \ddots & & & \\ & & & 0 & \mu_r & \\ & & & -\mu_r & 0 & \\ & 0 & & & & A_{2(n-r)} \end{array}\right)$$

(We call this the r-fold skew-diagonalization of A). If $A_{2(n-r)}$ is the null matrix the rest of the proof is trivial.

Suppose $A_{2(n-r)}$ is not the null matrix.

Let $K_{r+1} = \left(\begin{array}{c|c} I & \phi \\ \hline \phi & D_{r+1} \end{array}\right)$ be orthogonal, and

let

$$D_{r+1}^{T} A_{2(n-r)} D_{r+1} = \left(\begin{array}{cc|c} 0 & \mu_{r+1} & \\ -\mu_{r+1} & 0 & \phi \\ \hline \multicolumn{2}{c|}{\phi} & A_{2(n-r-1)} \end{array}\right)$$

The existence of D_{r+1} is guaranteed by Lemma 2.3.

Hence $K_{r+1}^{T} M_r^{T} A\ M_r K_{r+1} =$

$$\left(\begin{array}{cc|cc|c} 0 & \mu_1 & & & \\ -\mu_1 & 0 & & & 0 \\ \hline & & 0 & \mu_{r+1} & \\ & & -\mu_{r+1} & 0 & \\ \hline 0 & & & & A_{2(n-r-1)} \end{array}\right)$$

But $M_r K_{r+1}$ is a product of orthogonal matrices and so is again orthogonal. Let $M_{r+1} = M_r K_{r+1}$. Then we have shown that if there exists an orthogonal matrix M_r which is supposed to skew-diagonalize A to an r-fold level, there also exists

an orthogonal matrix M_{r+1} which skew-diagonalizes A to an (r+1)-fold level.

But M_1 exists. Hence M_r exists for all r positive and integral, where the matrix A has the nonzero eigenvalues $i\mu_1 . \; . \; . \; . \; i\mu_r$ ($r \leq n$). Let $V = M_1 M_2 . \; . \; . M_n$. Then V is orthogonal and

$$V_T A V = \begin{pmatrix} \begin{matrix} 0 & \mu_1 \\ -\mu_1 & 0 \end{matrix} & & & 0 \\ & \begin{matrix} 0 & \mu_2 \\ -\mu_2 & 0 \end{matrix} & & \\ & & \ddots & \\ 0 & & & \begin{matrix} 0 & \mu_n \\ -\mu_n & 0 \end{matrix} \end{pmatrix}$$

where some of the μ_i are zero. <u>This proves the theorem.</u>

<u>Corollary.</u> Let A be real skew-symmetric of order 2n+1. Then there exists an orthogonal matrix V such that

$$V_T A\ V = \left(\begin{array}{cc|ccc|c}
0 & \mu_1 & & & & \\
-\mu_1 & 0 & & & 0 & \\
\hline
 & & \ddots & & & \\
\hline
 & & & 0 & \mu_n & \\
 & 0 & & -\mu_n & 0 & \\
\hline
 & & & & & 0
\end{array}\right)$$

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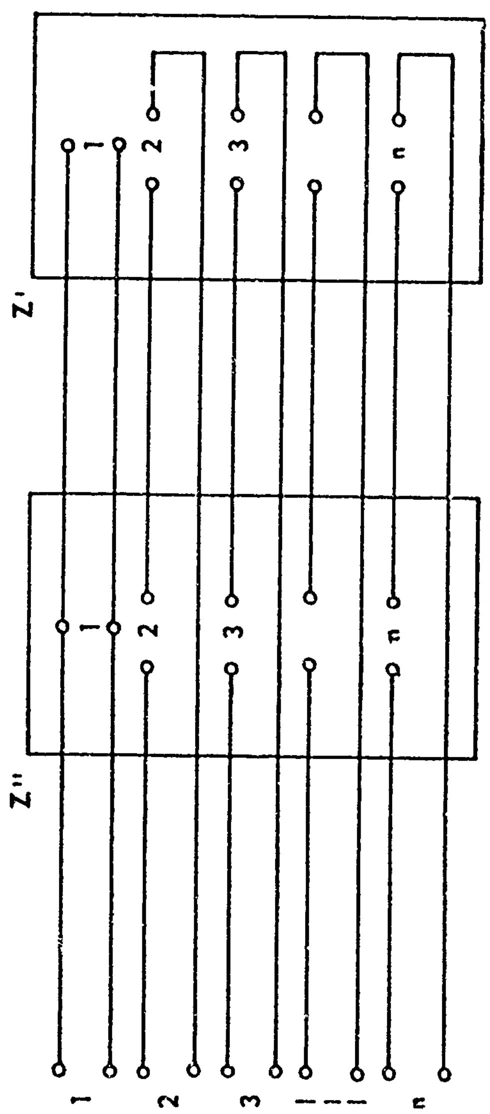


Figure 1 - Series Parallel Connection

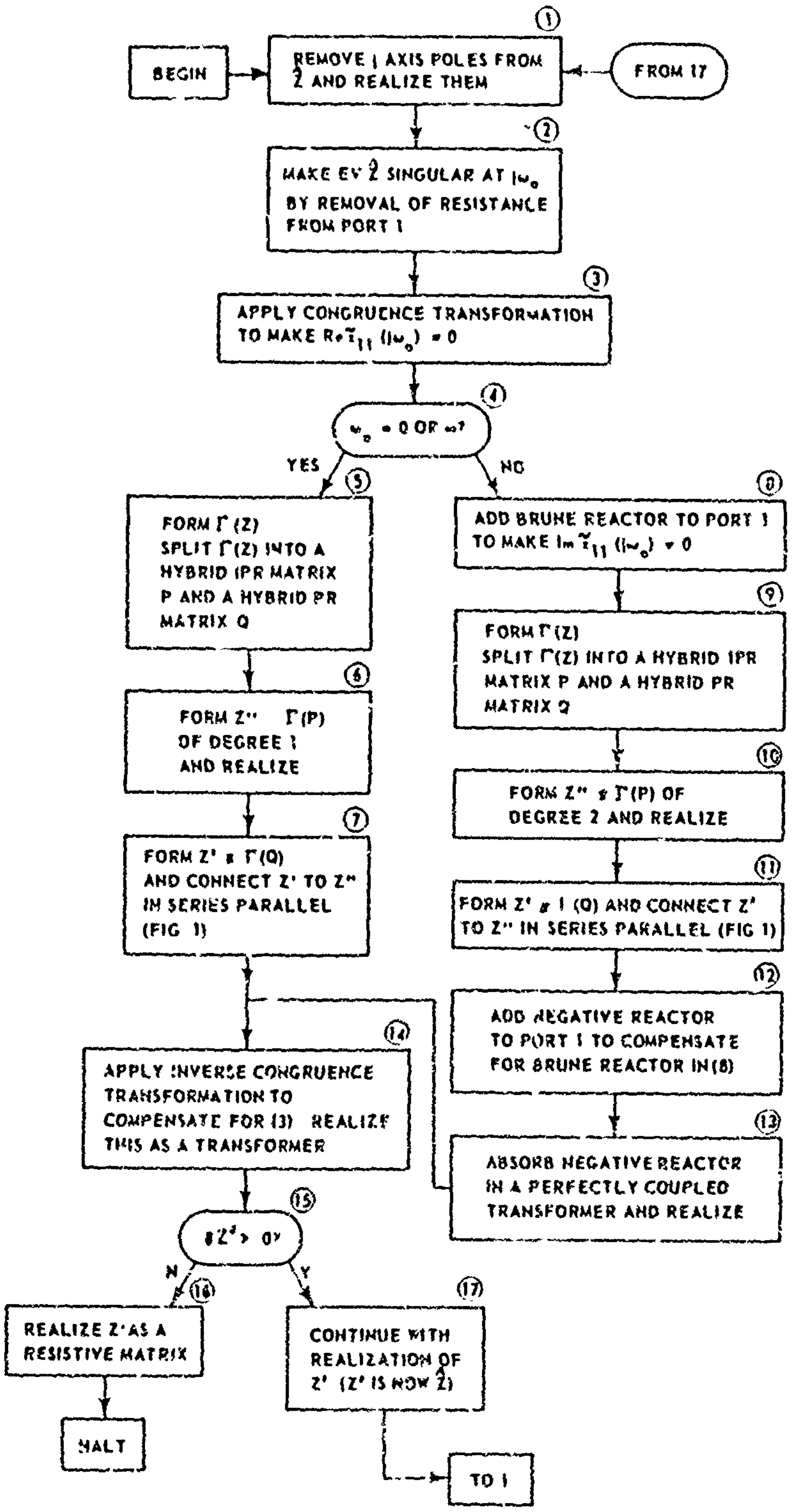


Figure 2 - Flow Cha   of Synthesis Procedure

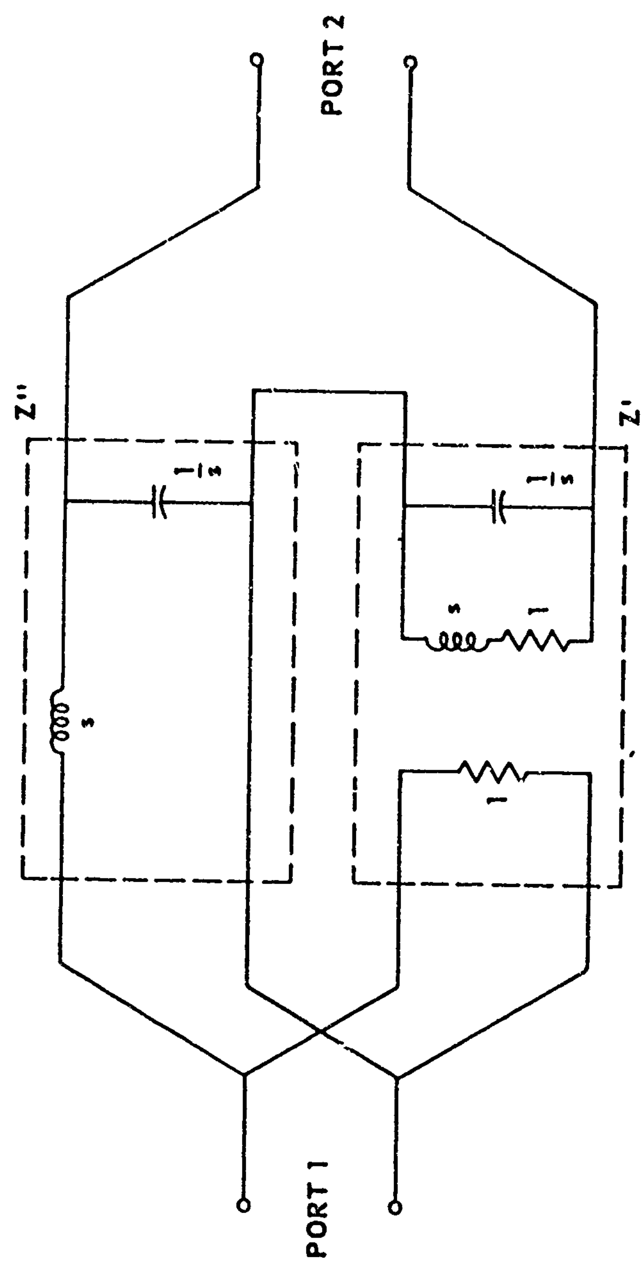


Figure 3 - Realization of Z(s) of Example 1

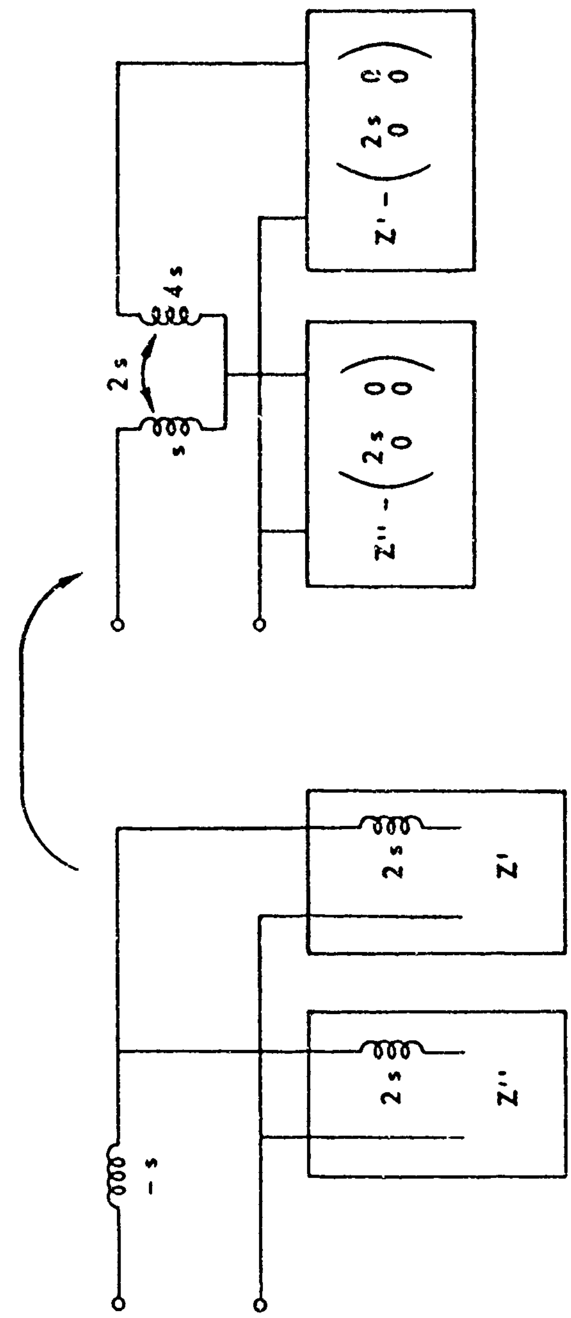


Figure 4 - Brune Cycle Transformer

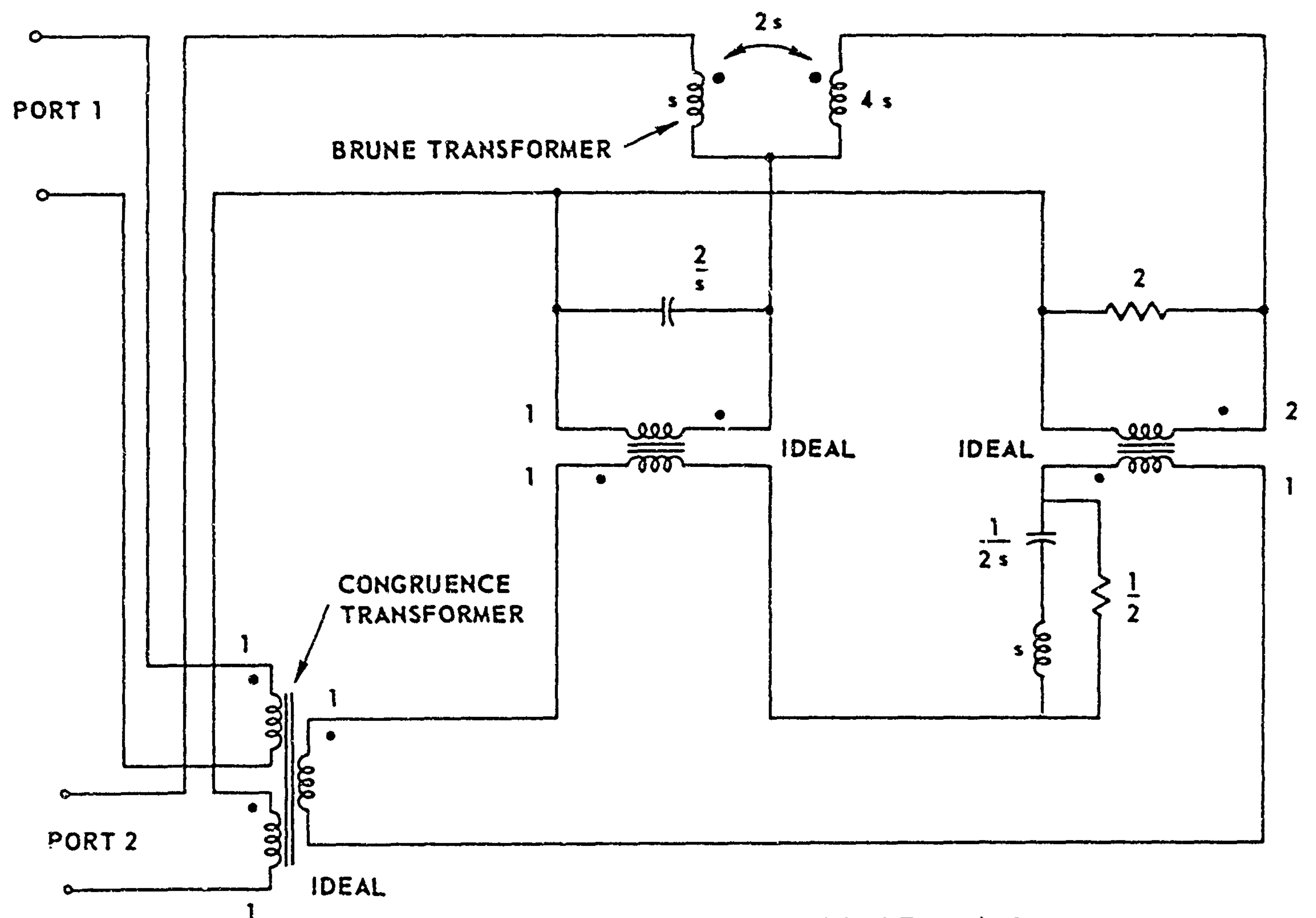


Figure 5 - Realization of Z(s) of Example 2

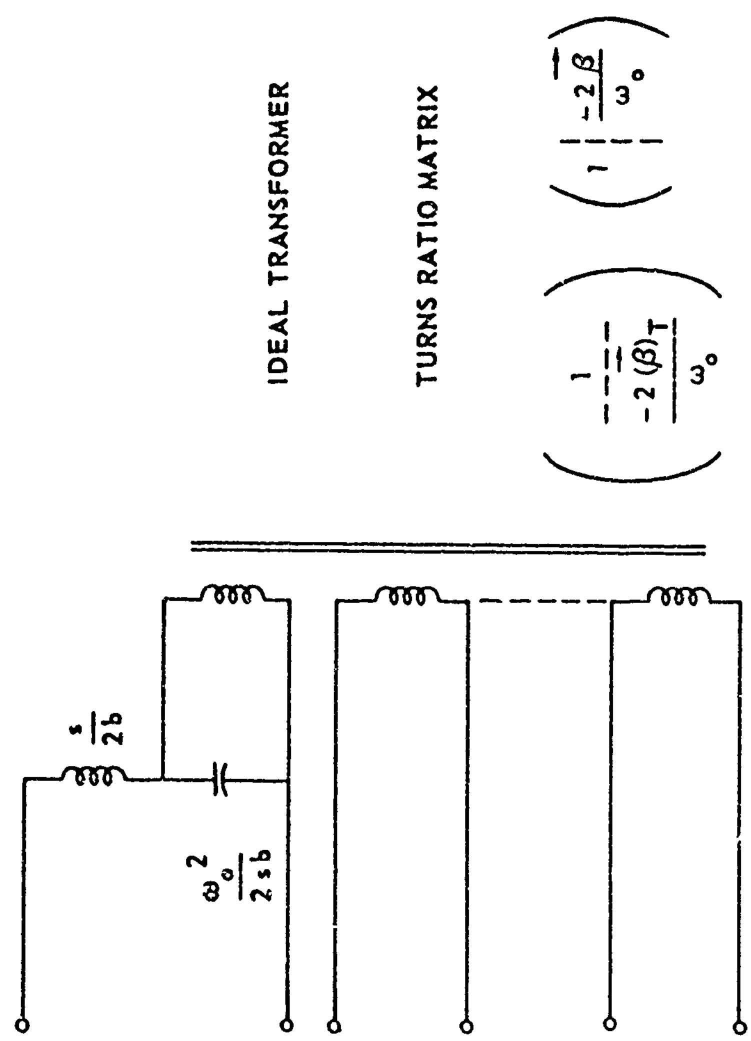
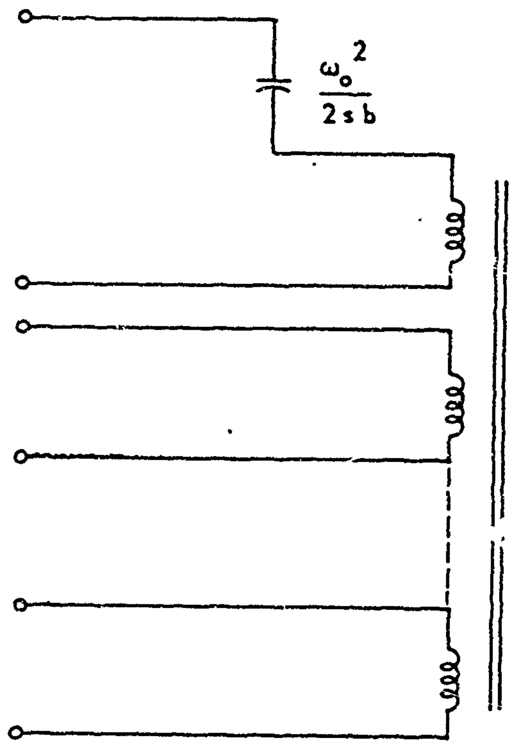


Figure 6 - Realization of Z''(s)



PERFECTLY COUPLED TRANSFORMER

IMPEDANCE MATRIX

$$\frac{s}{2b}\begin{pmatrix} 1 \\ \hline \frac{2(\vec{\beta})_T}{\omega_o} \end{pmatrix}\begin{pmatrix} 1 & \vline & \frac{2\vec{\beta}}{\omega_o} \end{pmatrix}$$

Figure 7 - Alternate Realization of Z''

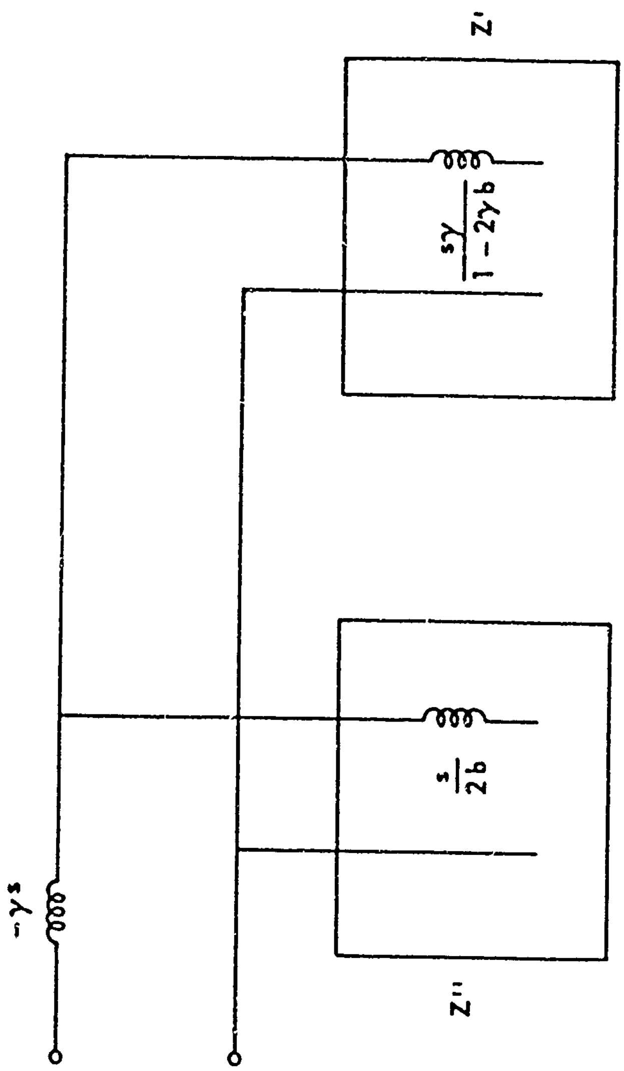


Figure 8 - Brune Cycle Realization

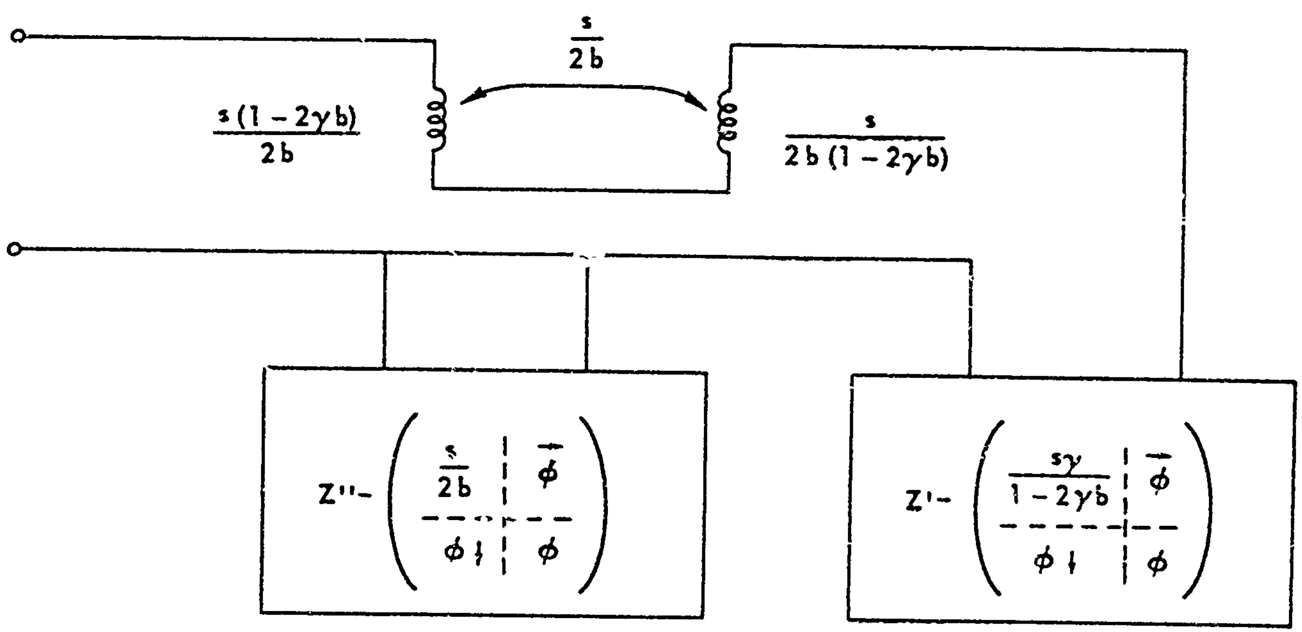


Figure 9 - Transformer Realization of Brune Cycle

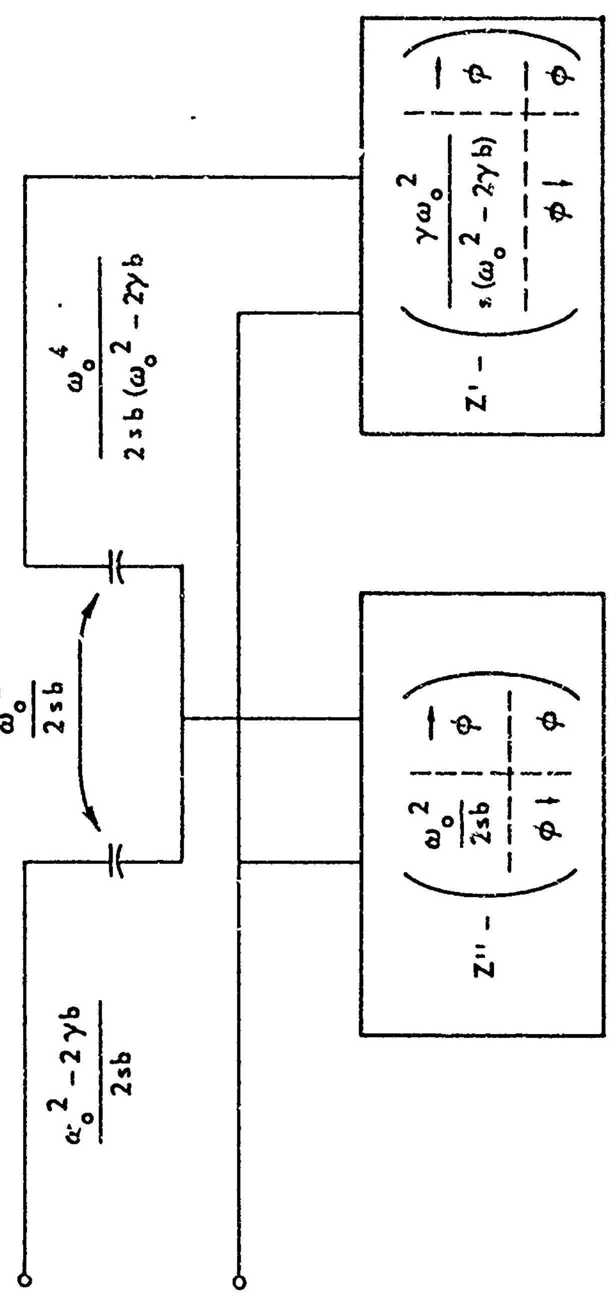


Figure 10 — Brune Cycle Capacitive Transformer

Figure 11 is deleted.

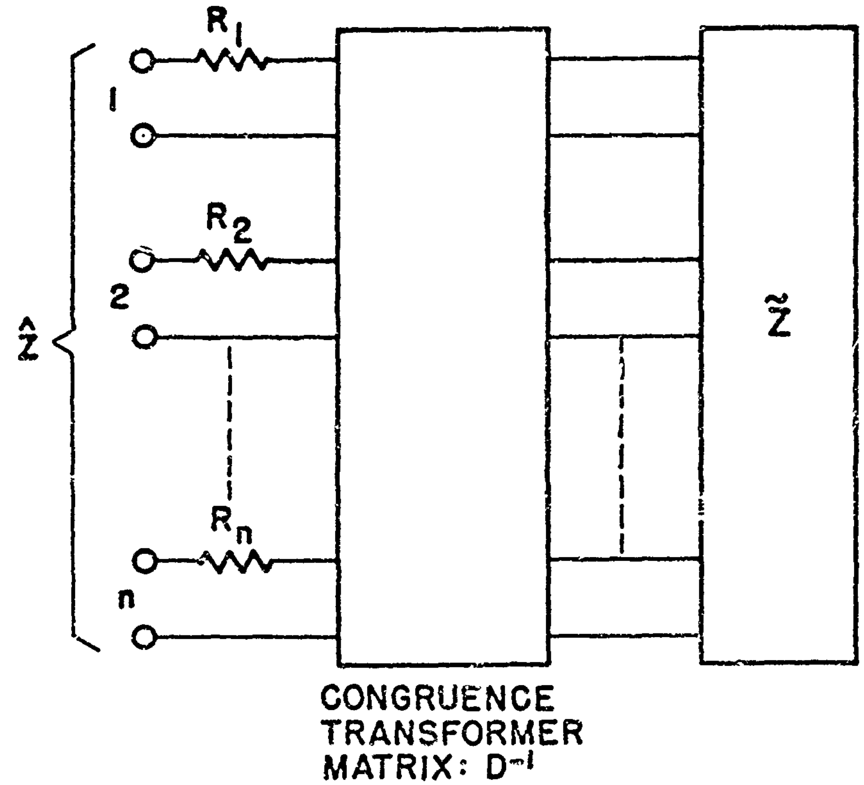


FIGURE 12– CONDITIONING $\hat{Z}$ FOR SYNTHESIS

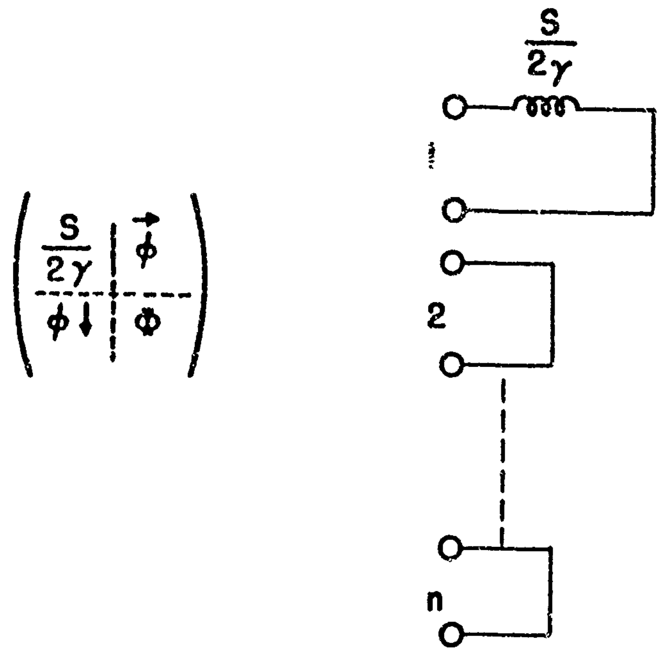


FIGURE 13— SYNTHESIS OF THE FIRST PART OF Z″

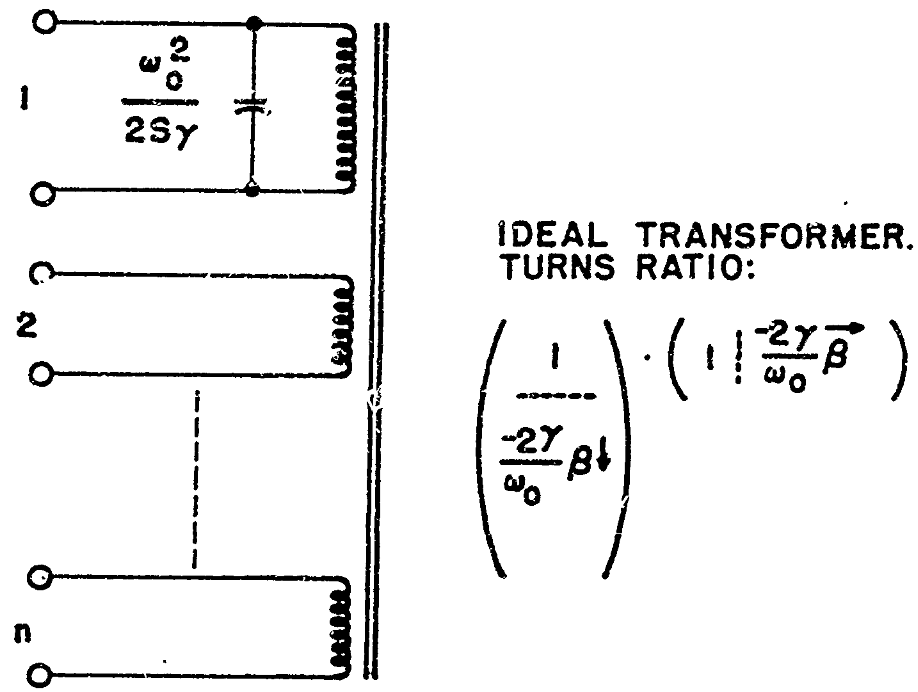


FIGURE 14–SYNTHESIS OF SECOND PART OF Z"

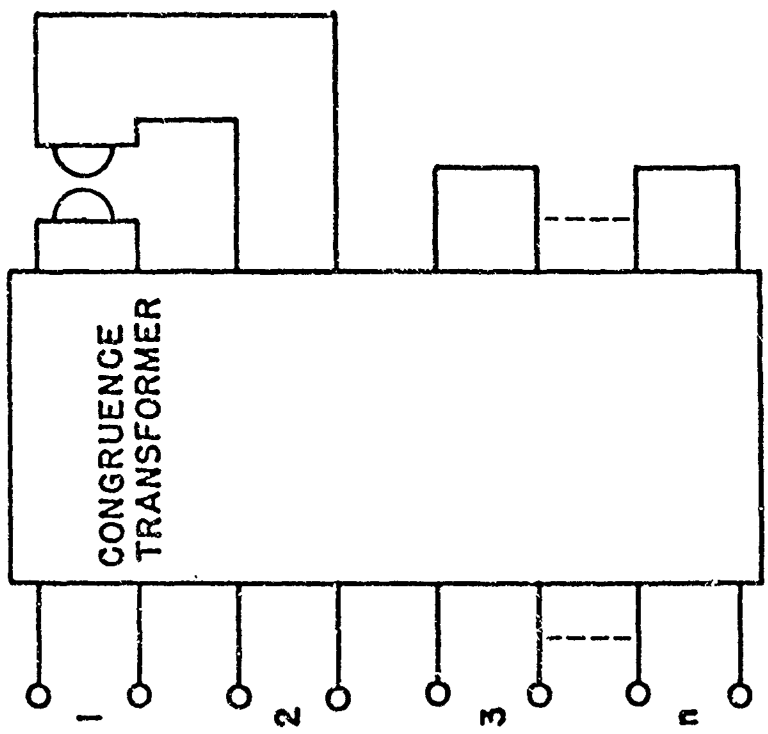


FIGURE 15—SYNTHESIS OF THIRD PART OF Z″

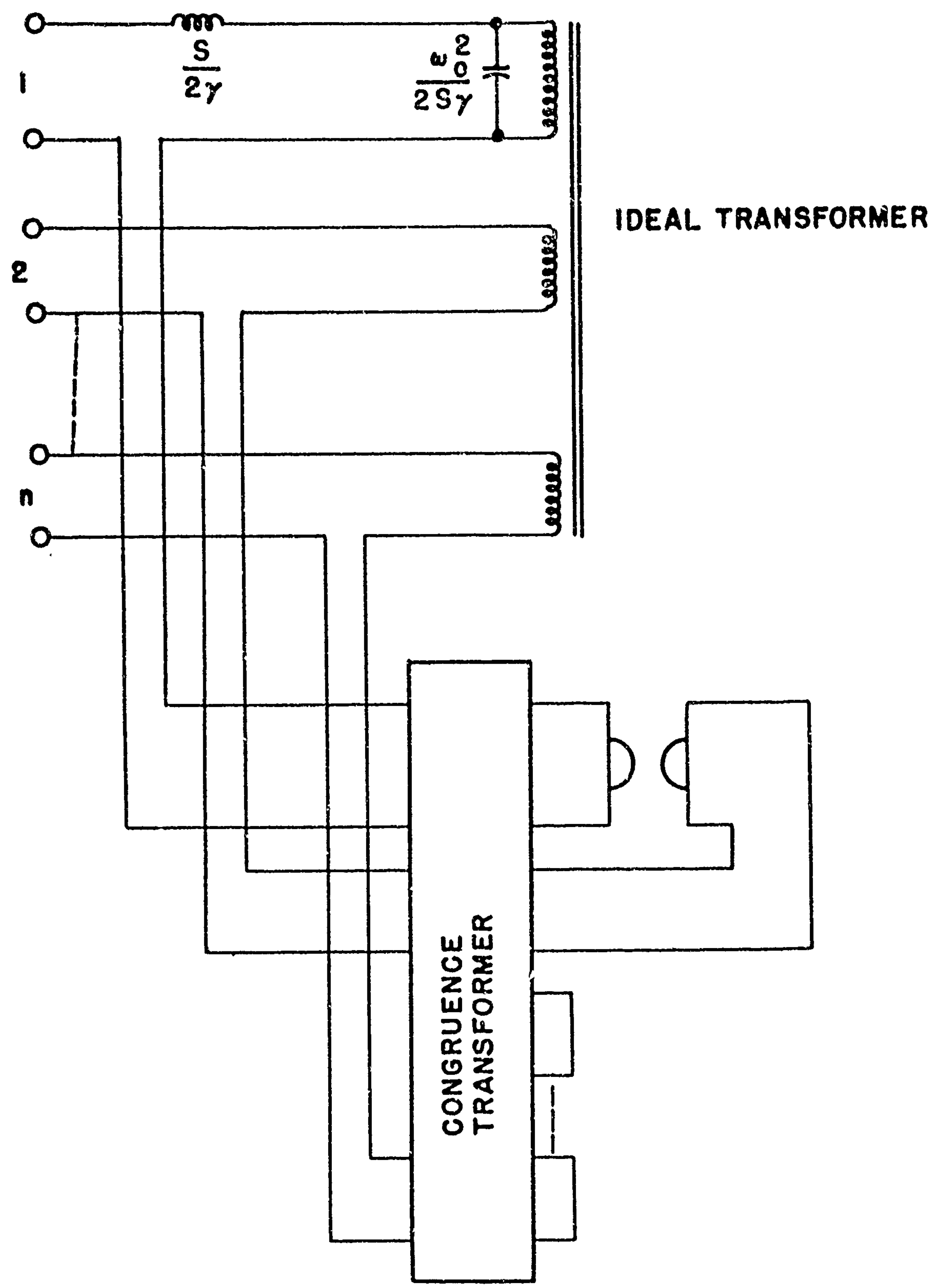


FIGURE 16— COMPLETE SYNTHESIS OF Z″

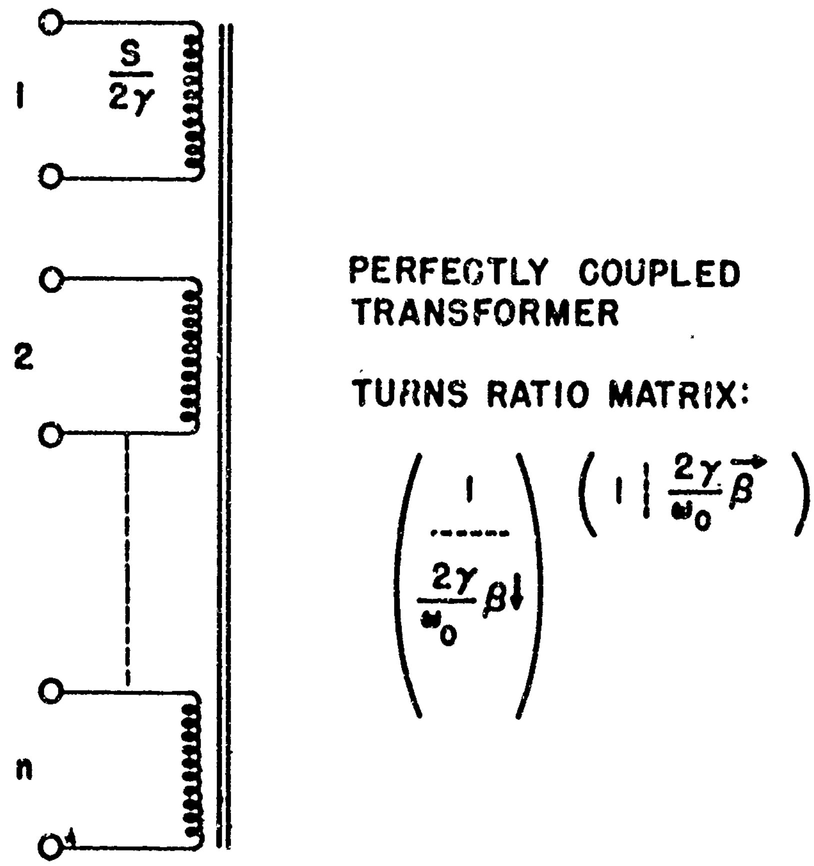


FIGURE 17— SECOND TERM OF Z" (ALTERNATE REALIZATION)

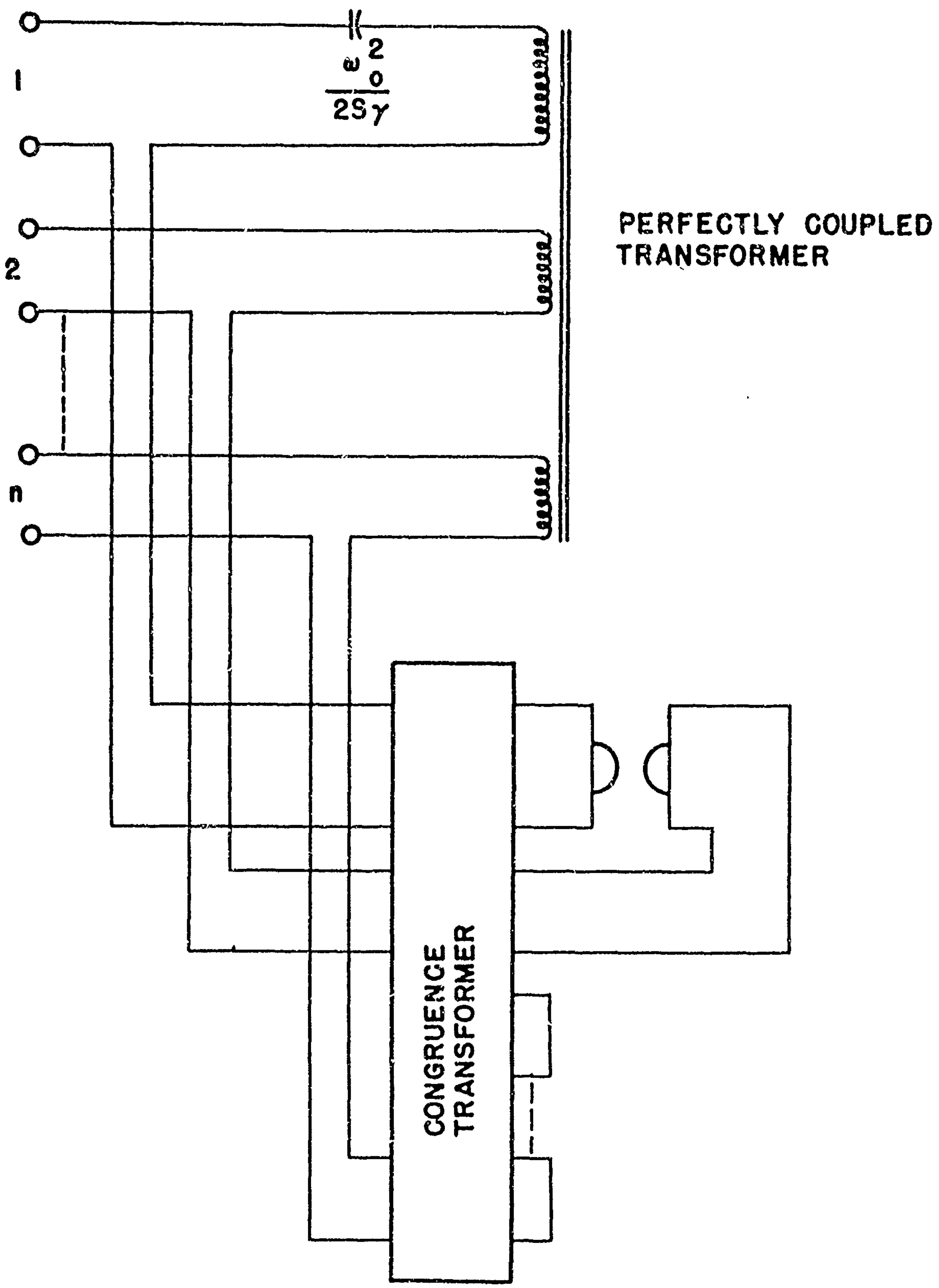


FIGURE 18- ALTERNATE REALIZATION OF Z''

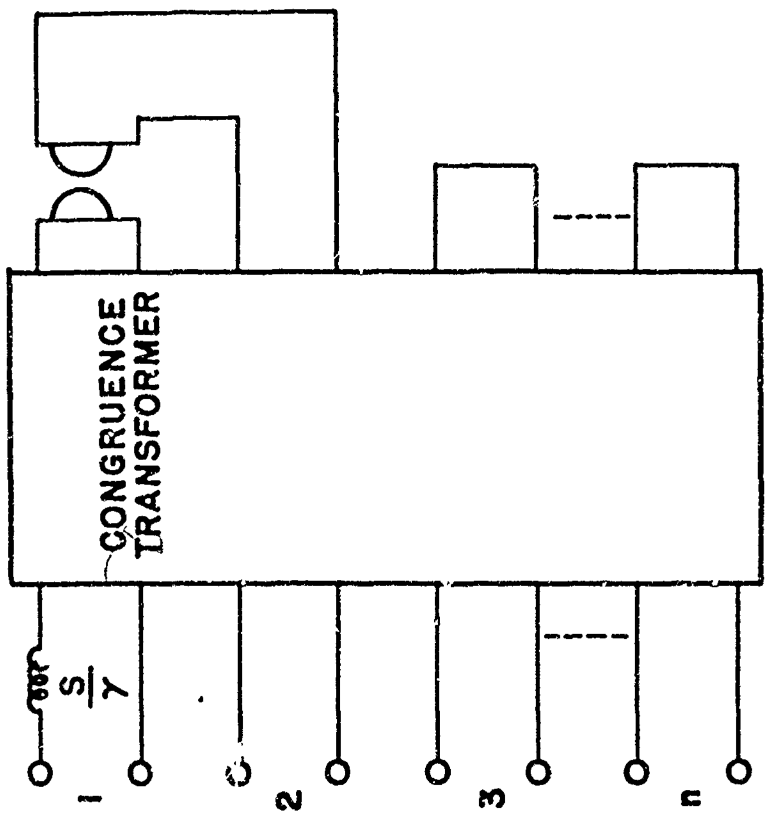


FIGURE 19— REALIZATION OF Z'' WHEN $\omega_0 = 0$

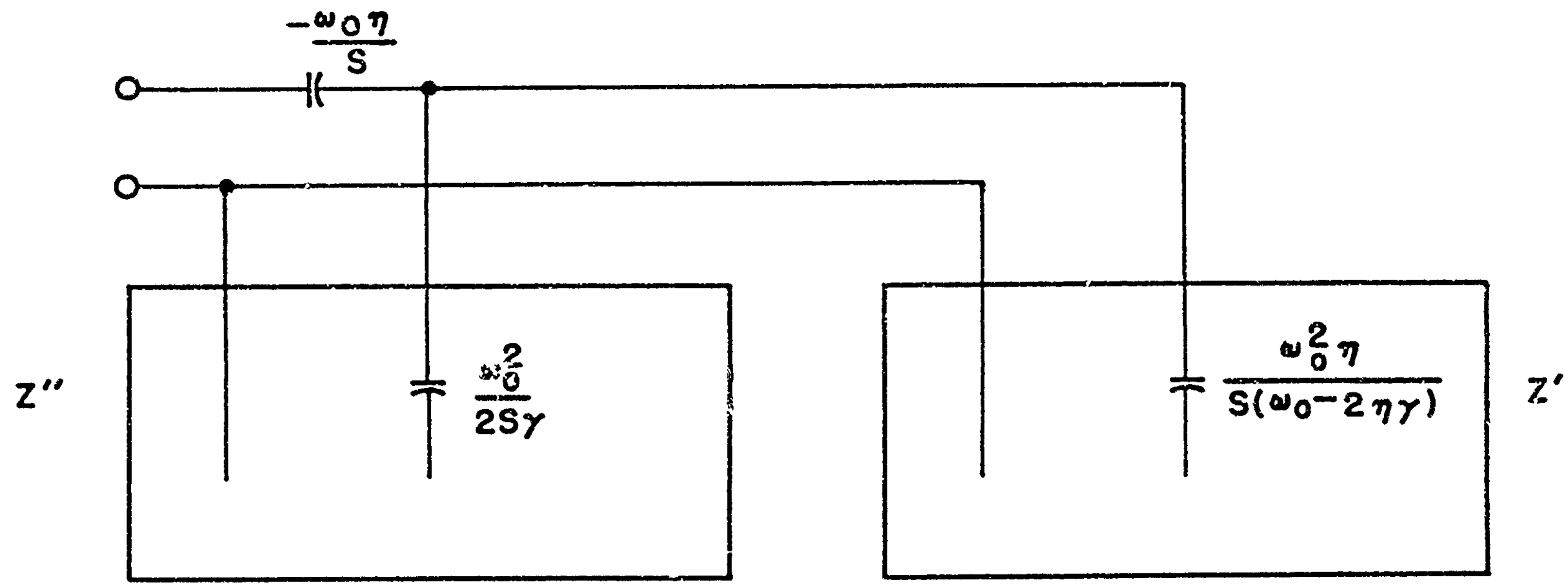


FIGURE 20— COMPENSATION FOR THE ADDED BRUNE REACTOR

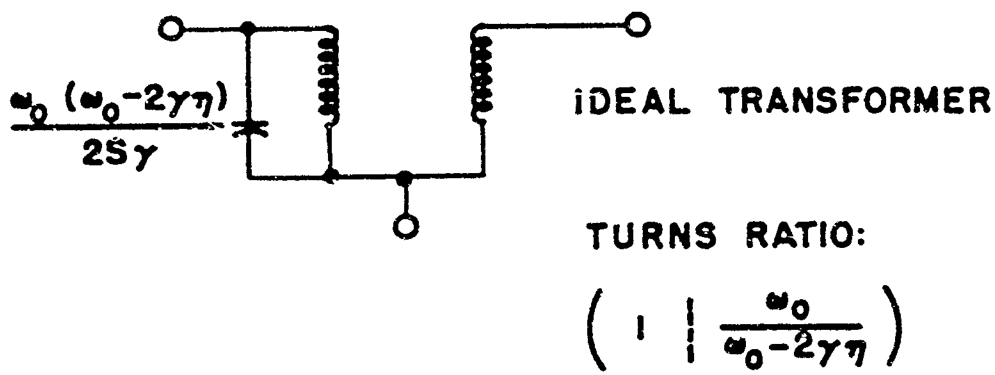


FIGURE 21 – REALIZATION OF T (CAPACITIVE CASE)

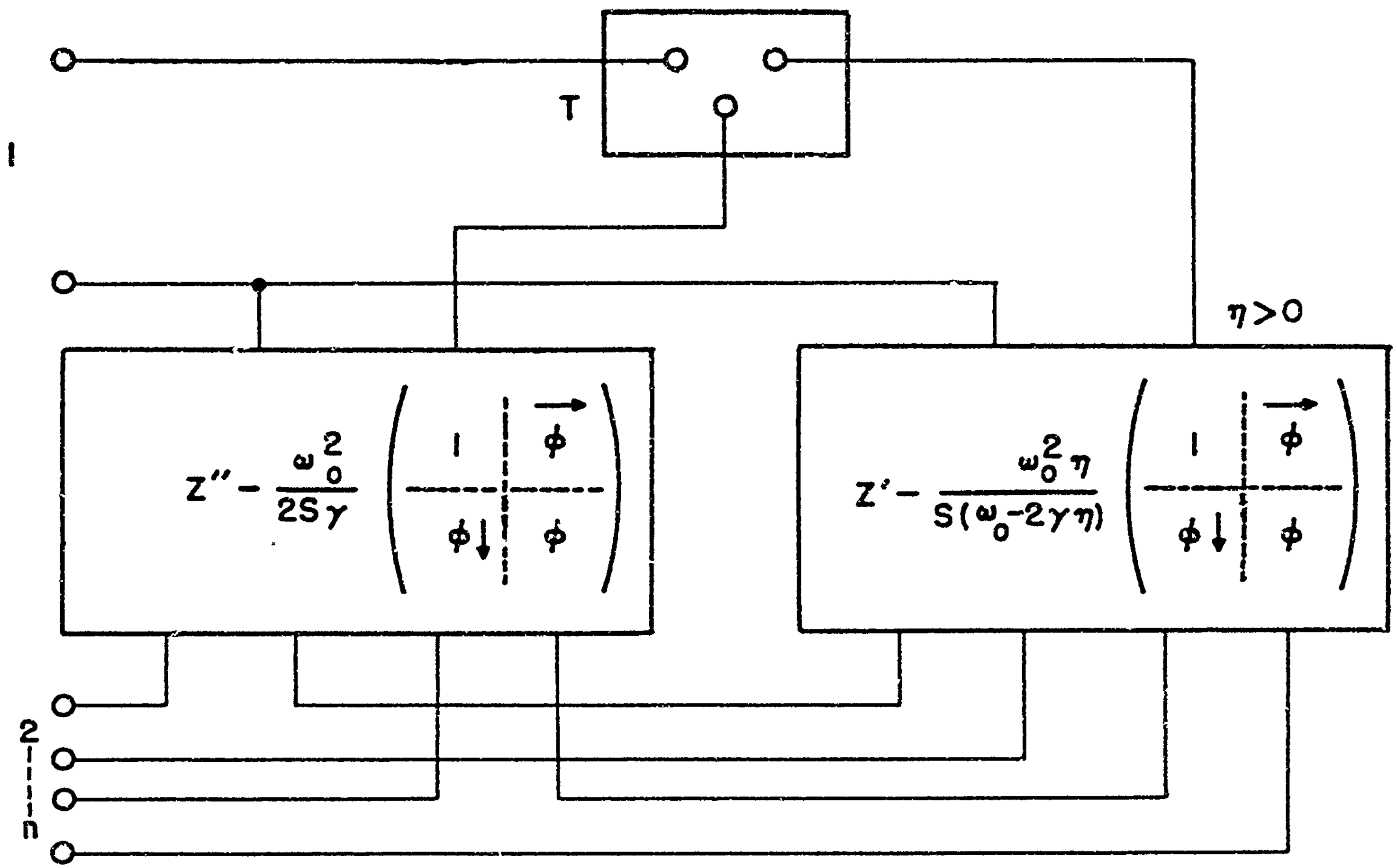


FIGURE 22— REALIZATION OF $\tilde{Z}$ (CAPACITIVE CASE)

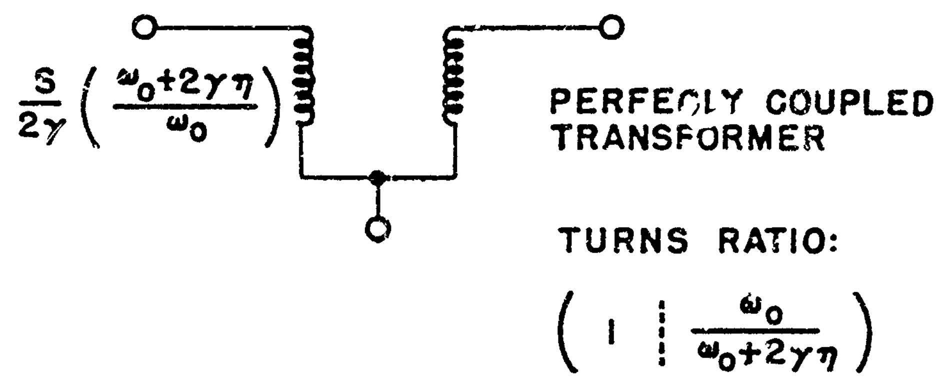


FIGURE 23— REALIZATION OF T(INDUCTIVE CASE)

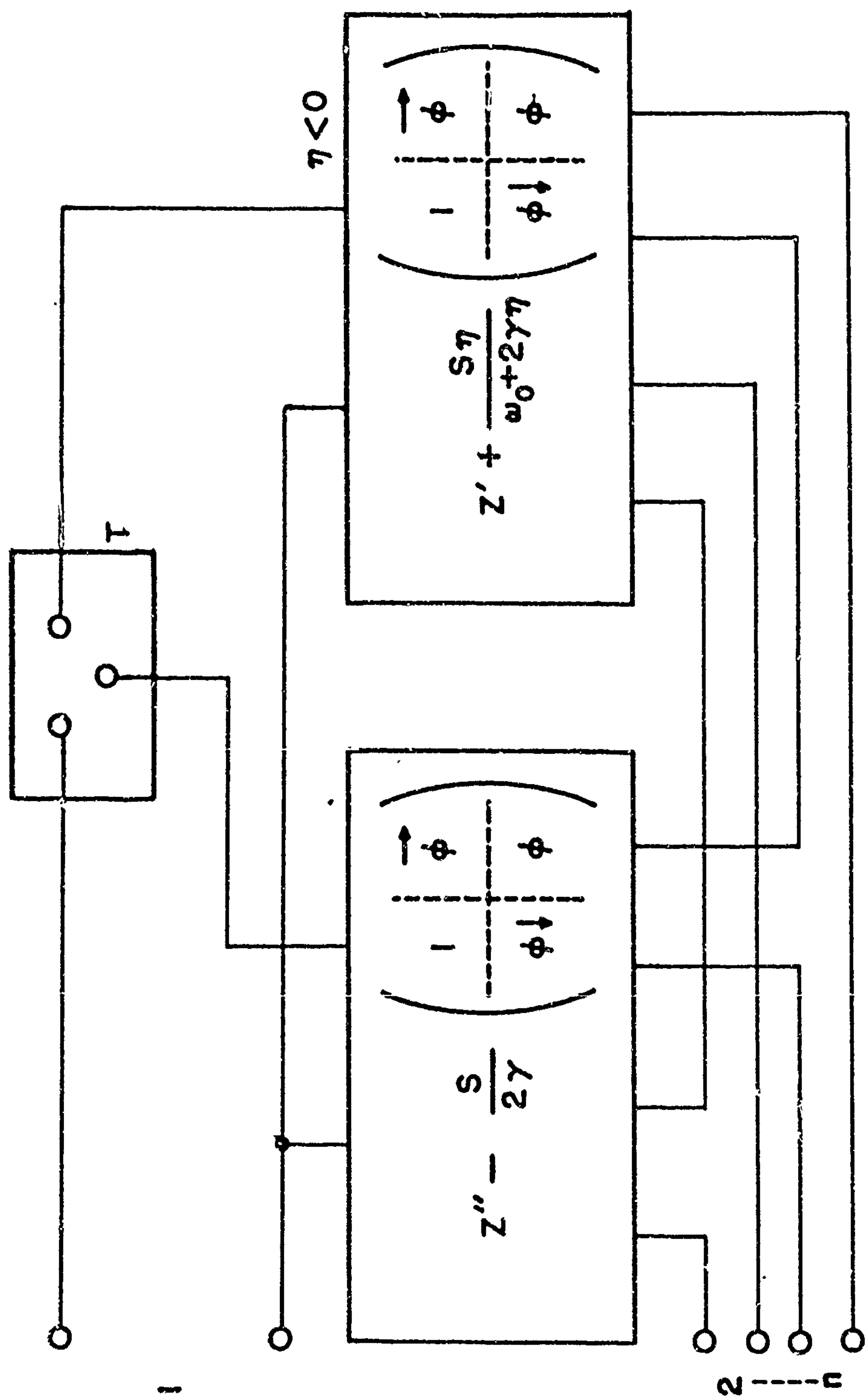


FIGURE 24—REALIZATION OF $\tilde{Z}$ (INDUCTIVE CASE)

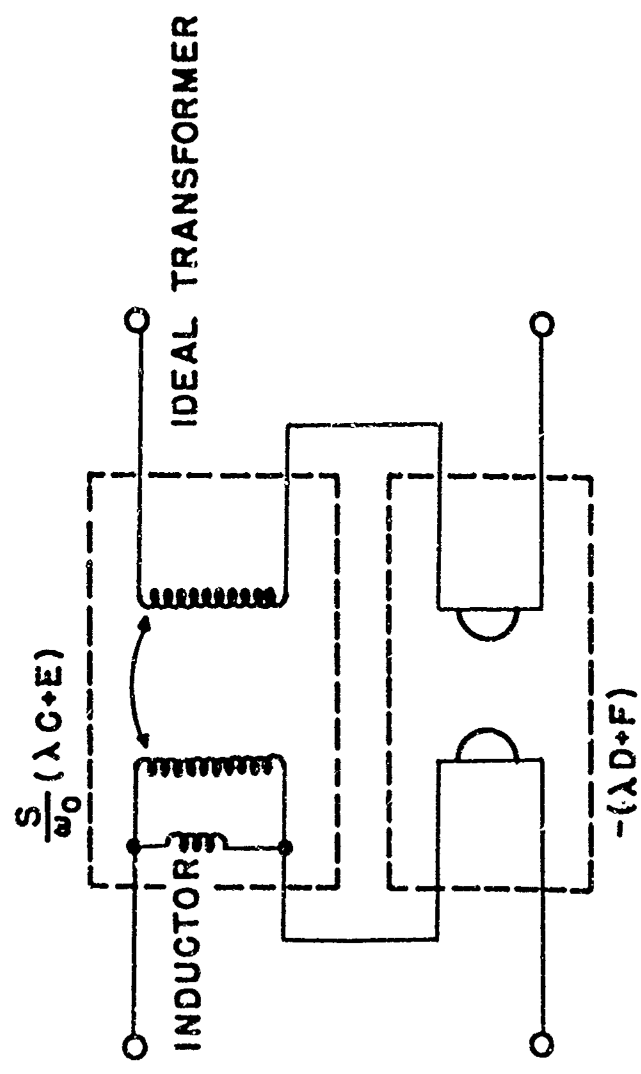


FIGURE 25— REALIZATION OF B

PORTS 1 AND 2 IN PARALLEL
PORTS 3 THROUGH n IN SERIES

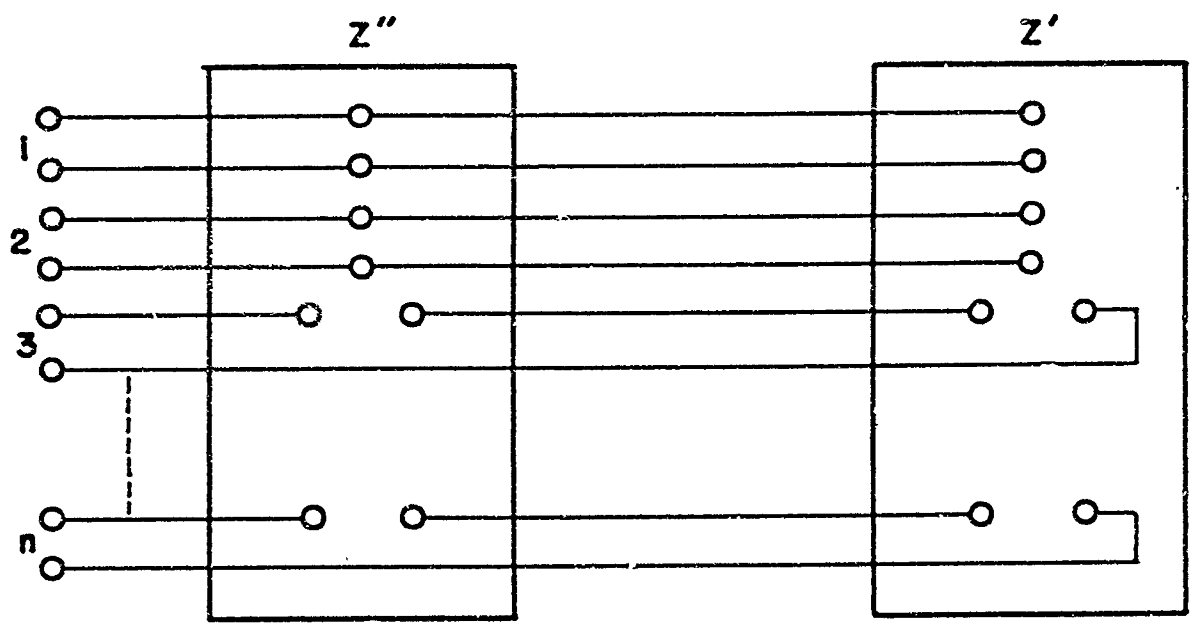


FIGURE 26— DECOMPOSITION OF Z

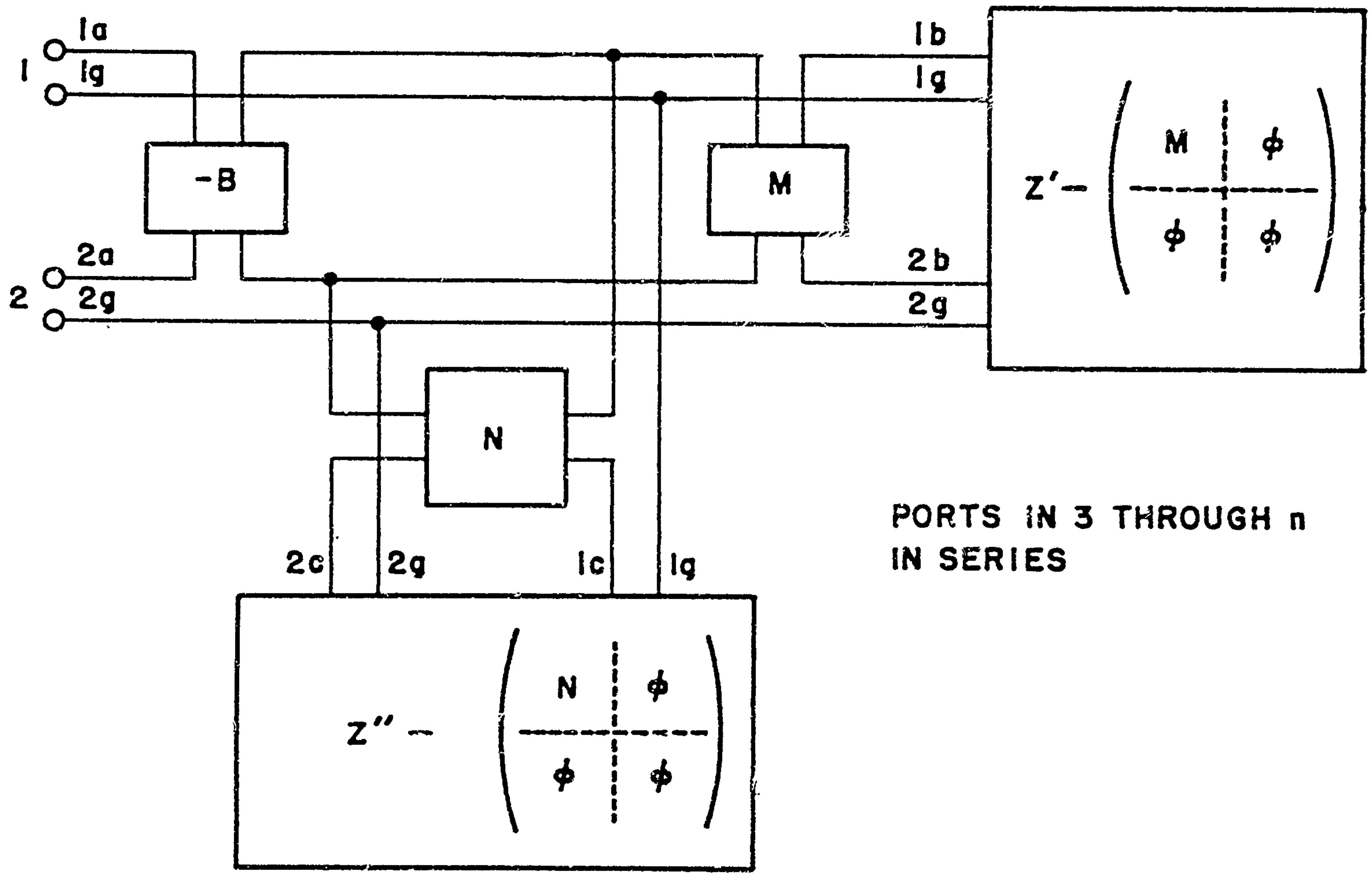


FIGURE 27— COMPENSATING FOR THE BRUNE SECTION

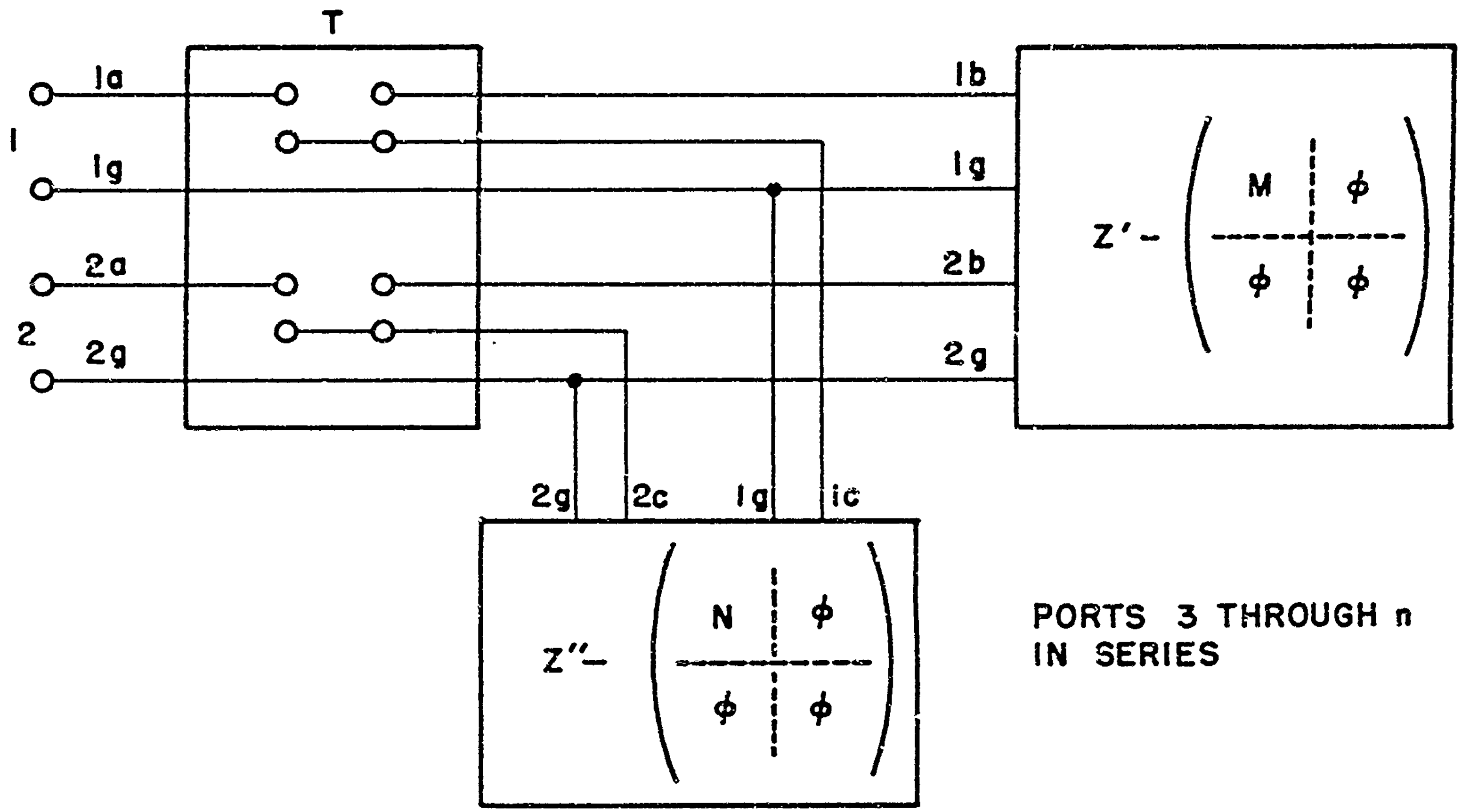


FIGURE 28— REALIZATION OF $\tilde{Z}$

$$Z = \tilde{Z}(S) + \left(\begin{array}{c|c} B(S) & \phi \\ \hline \phi & \phi \end{array}\right)$$

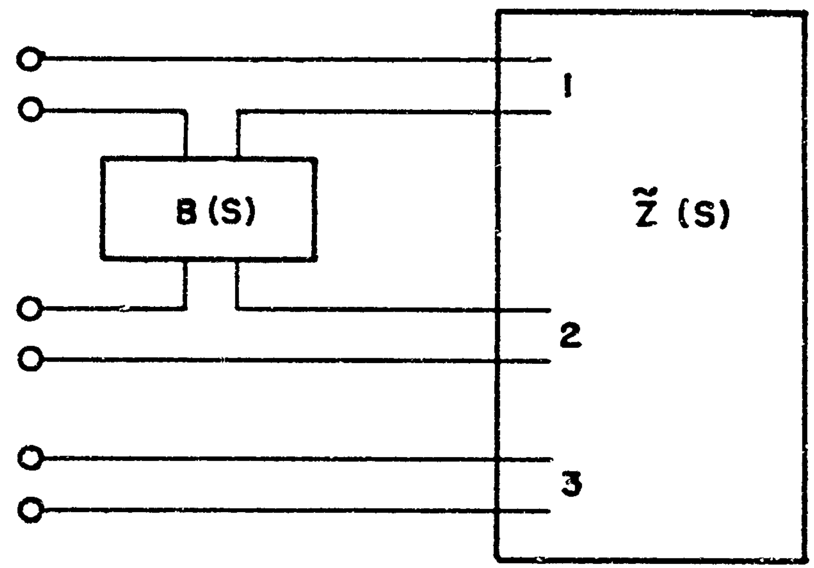


FIGURE 29— ADDITION OF B TO $\tilde{Z}$

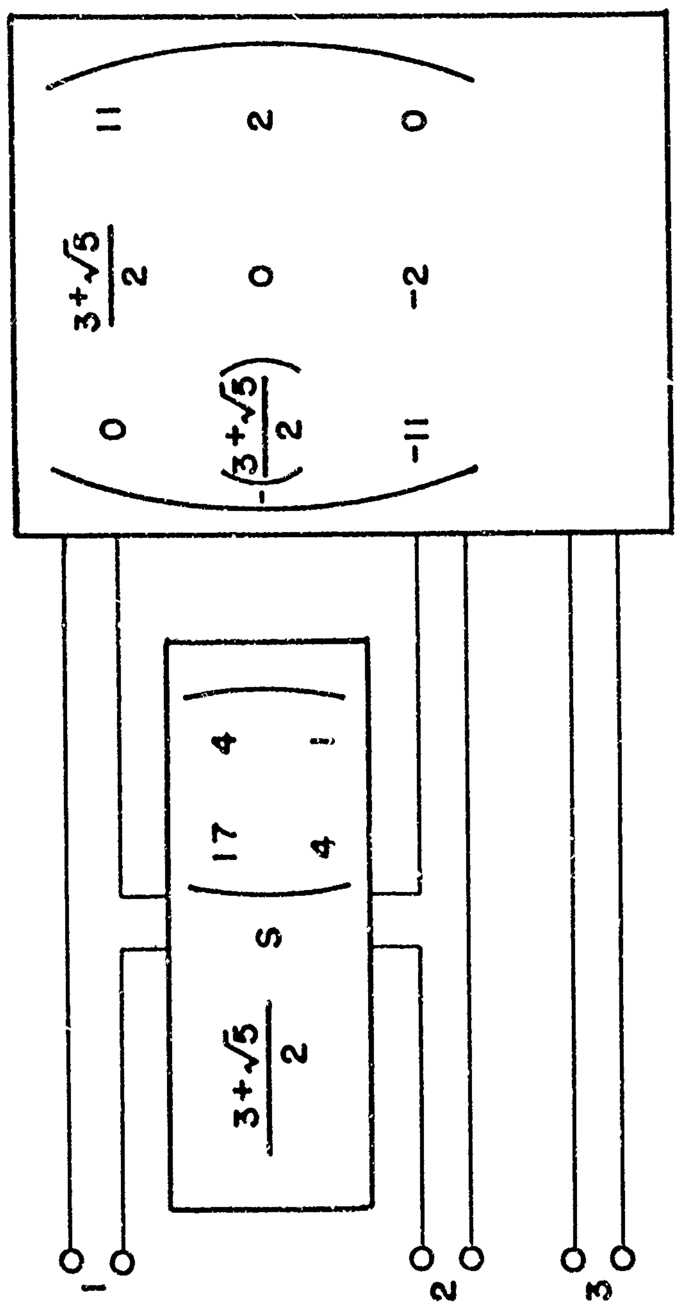

0
$\frac{3+\sqrt{5}}{2}$
11
$-\left(\frac{3+\sqrt{5}}{2}\right)$
0
2
-11
-2
0
$\frac{3+\sqrt{5}}{2}$
S
17
4
4
1
1
2
3

FIGURE 30— Z''(S)

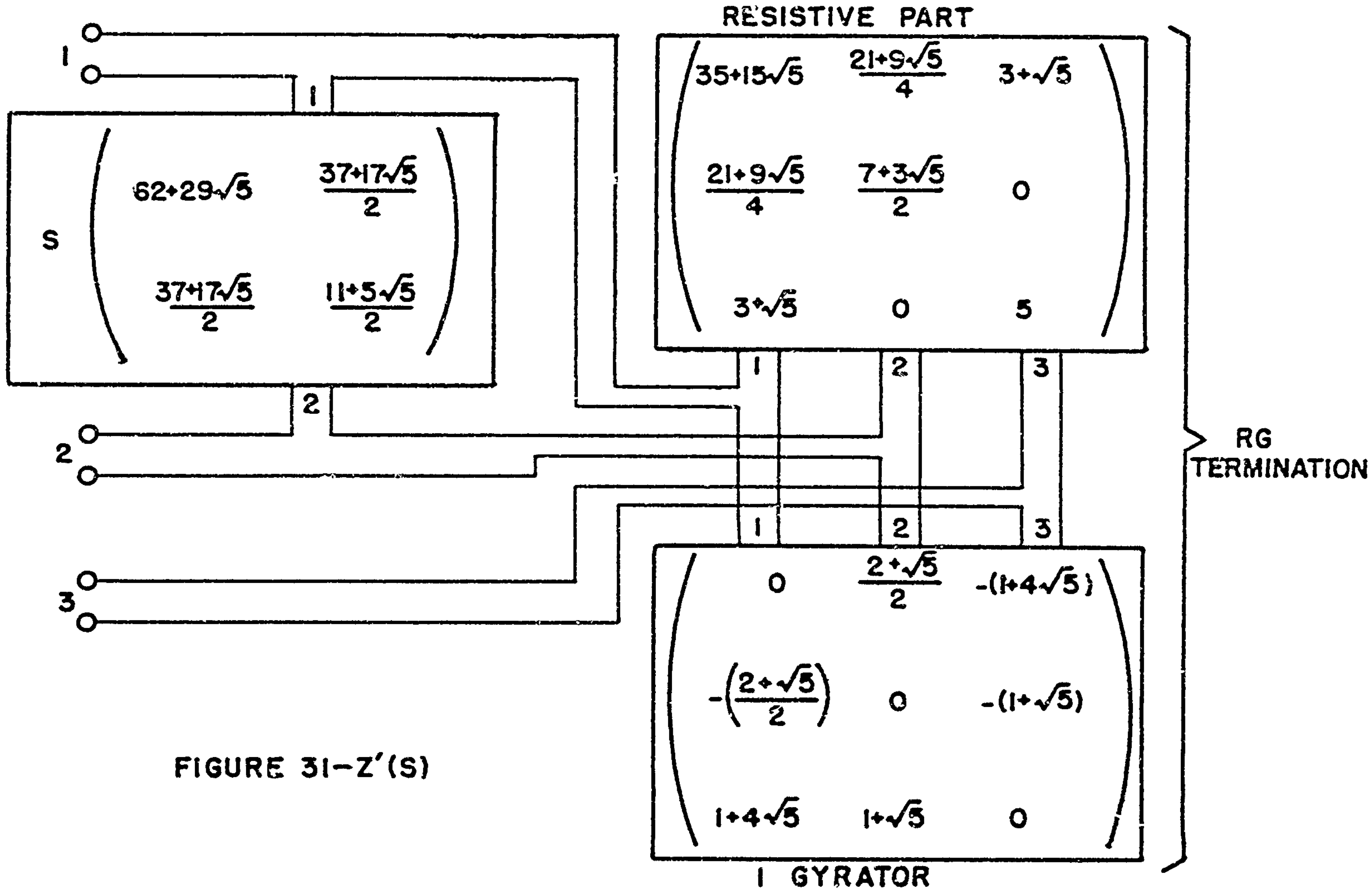

RESISTIVE PART
35+15√5
21+9√5 / 4
3+√5
21+9√5 / 4
7+3√5 / 2
0
3+√5
0
5
S
62+29√5
37+17√5 / 2
37+17√5 / 2
11+5√5 / 2
RG TERMINATION
0
2+√5 / 2
-(1+4√5)
-((2+√5)/2)
0
-(1+√5)
1+4√5
1+√5
0
1 GYRATOR
1
2
3

FIGURE 31—Z'(S)

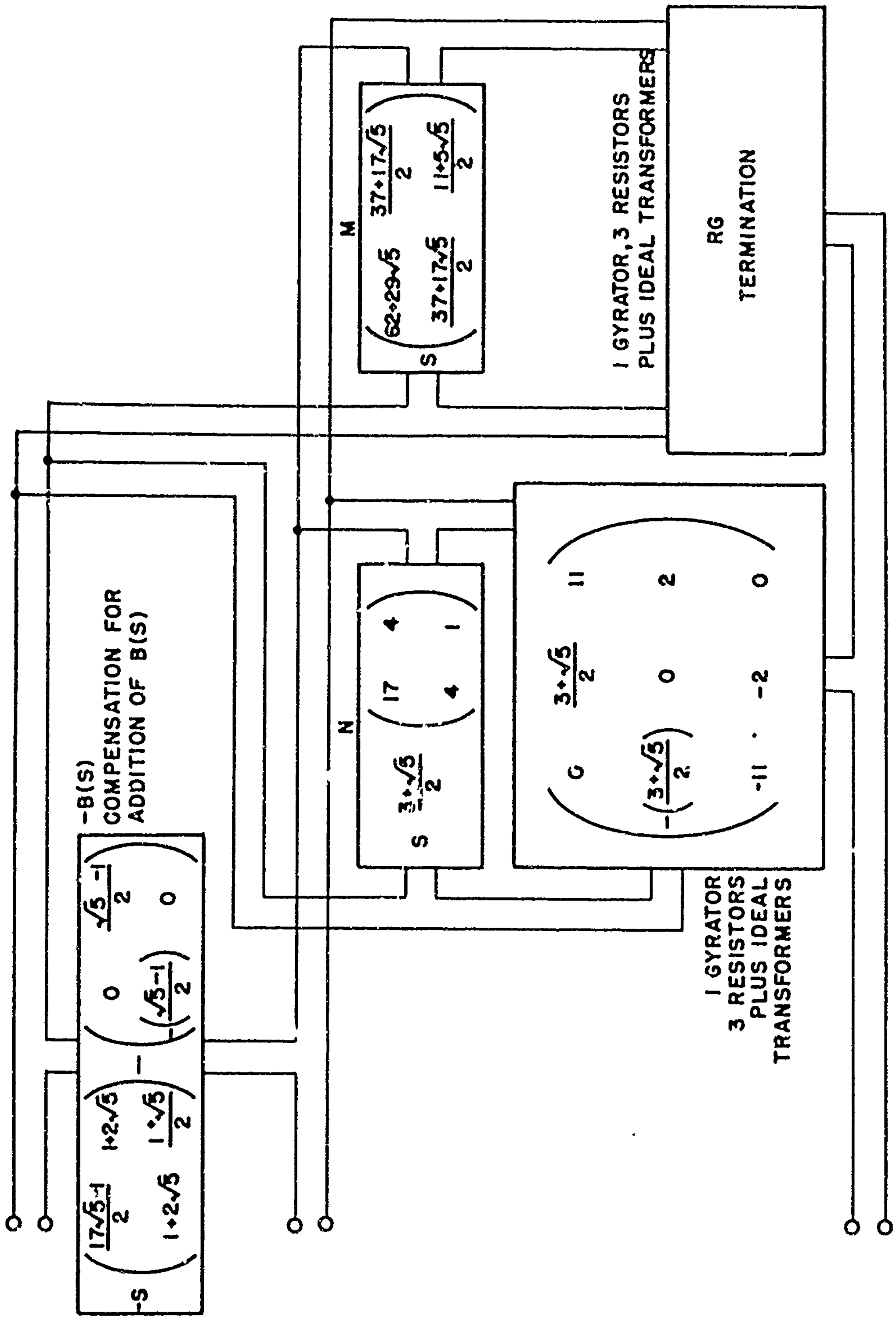


FIGURE 32- Z(S)-B(S)

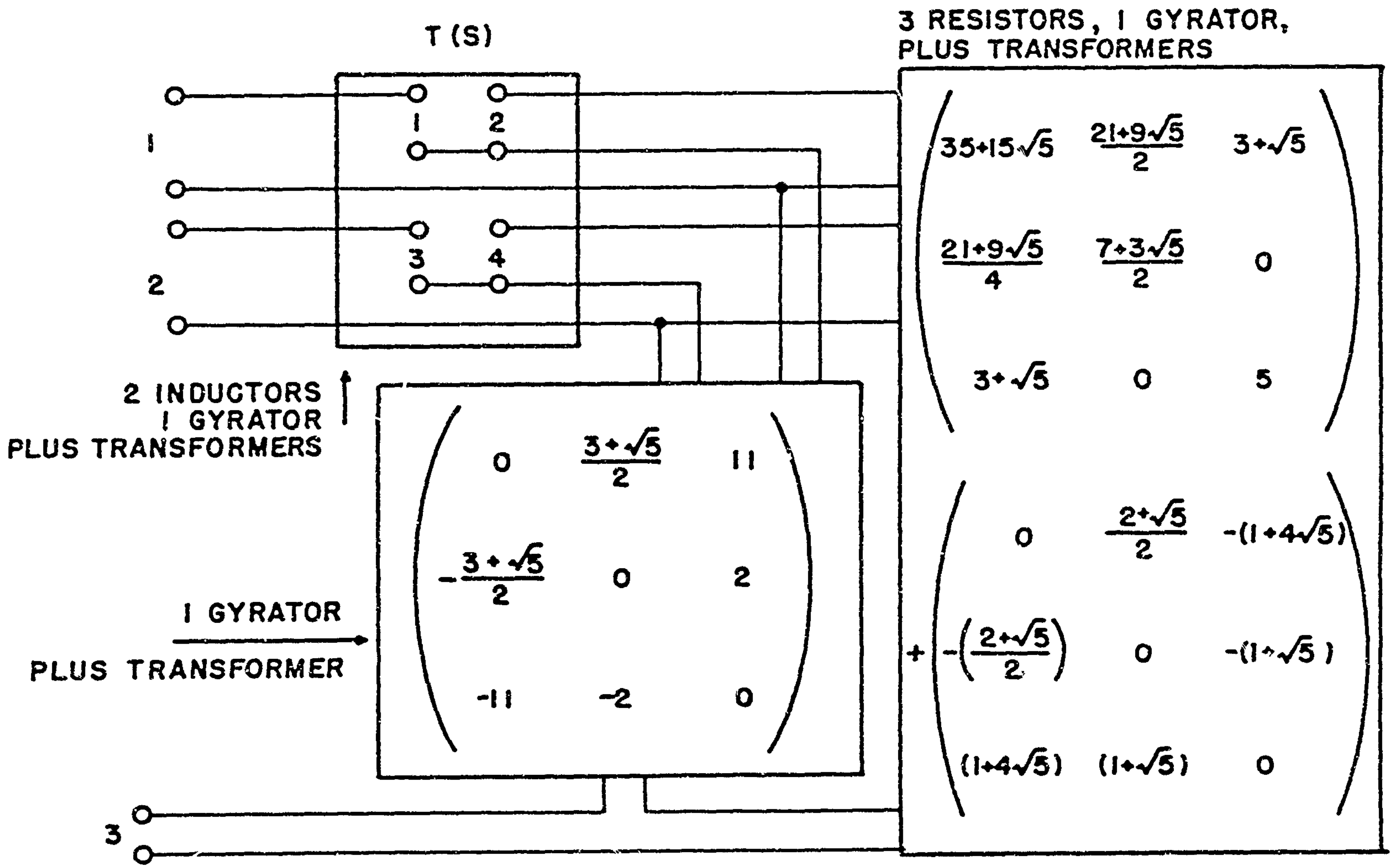


FIGURE 33— REALIZATION OF $\tilde{Z}$

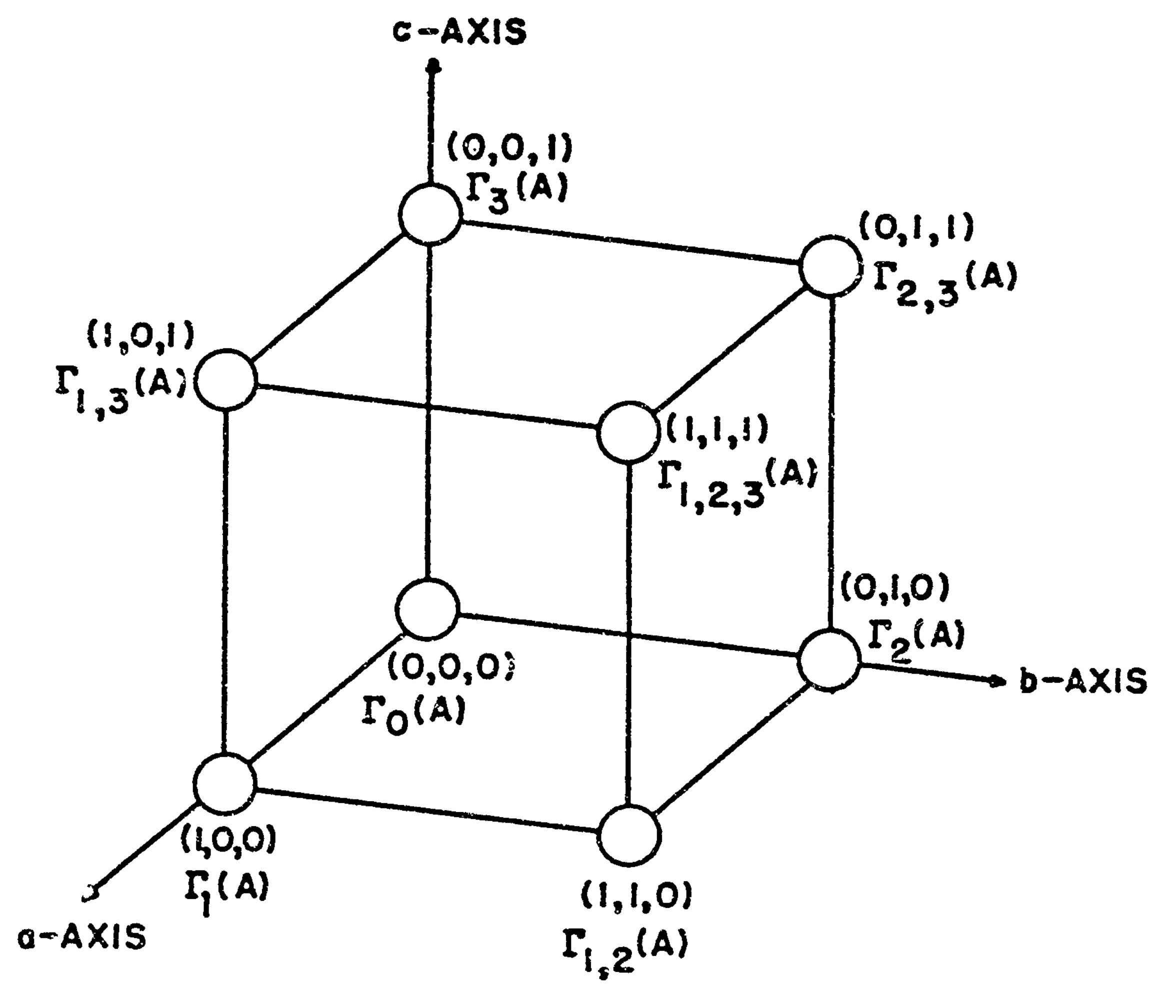


FIGURE 34— HYPERCUBE OF GYRATION MATRICES

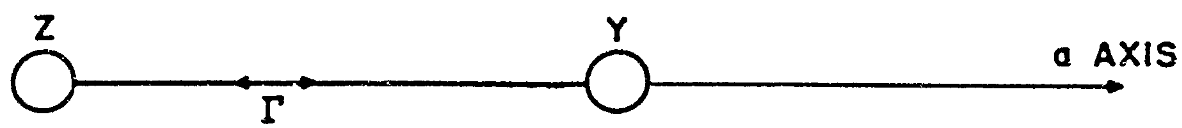


FIGURE 35— HYPERCUBE FOR $Z = \frac{S^2+1}{S}$

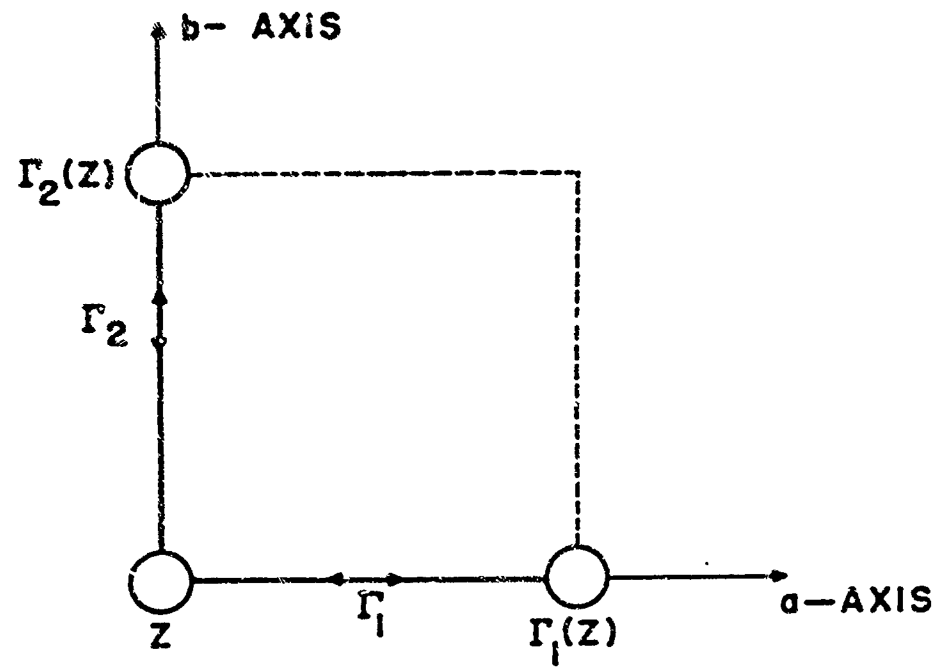


FIGURE 36— HYPERCUBE FOR $Z = S\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

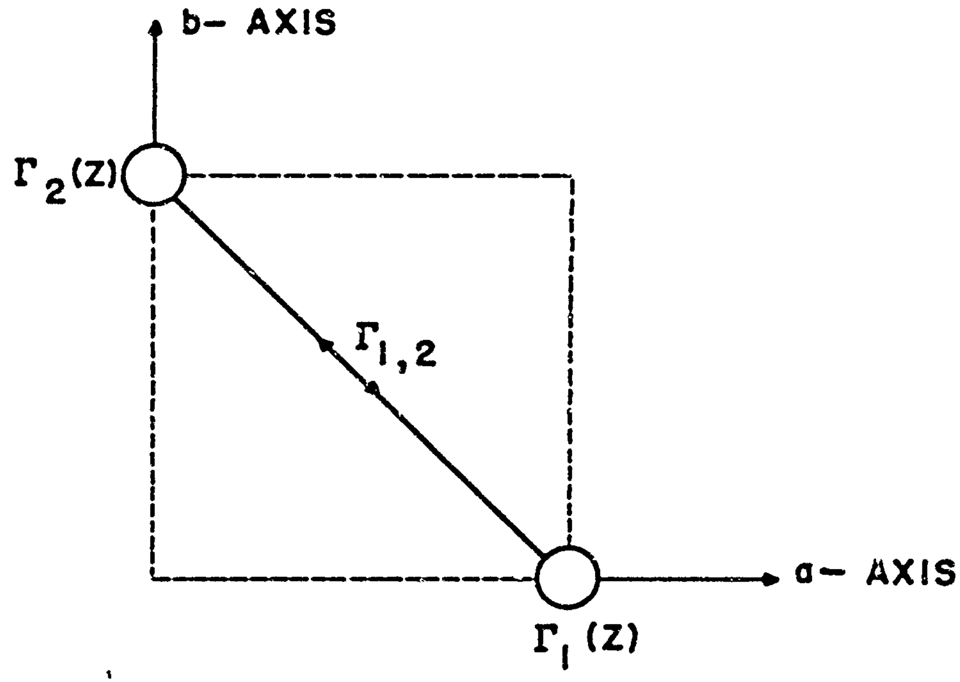


FIGURE 37- HYPERCUBE FOR $\Gamma_1(Z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$